Symmetries and Group Theory

Continuous Groups

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Contents

Abstract: A widely accepted bit of wisdom among Physicists is that symmetry implies degeneracy, and the larger the symmetry, the larger the degeneracy. What works forward ought to work backward (Newton's Third Law): if the degeneracy is greater than expected, the symmetry is greater than apparent.

- Taken from some book ...

Chapter 1

To Lie or not to Lie

A first look into Lie Groups and Lie Algebras

1.1 Lie group: general concepts

▶ Lie groups are important objects in Mathematics, Physics, ..., as they capture two very important areas of mathematics: algebra and geometry.

▶ The **algebraic properties** of a Lie group originate in the axioms for a group:

★ **Definition:** A set of g_i, g_j, $g_k, \ldots$ (called group elements or group operations) together with a combinatorial operation $\circ$ (called group multiplication) form a group G if the following axioms are satisfied:

(i) **Closure:** if $g_i \in G$, $g_j \in G$, then $g_i \circ g_j \in G$;

(ii) **Associativity:** $g_i, g_j, g_k \in G$ then $(g_i \circ g_j) \circ g_k = g_i \circ (g_j \circ g_k)$

(iii) **Identity:** There is an operator e (the identity operator) with the property that $\forall g_i \in G$ we get $g_i \circ e = g_i = e \circ g_i$;

(iv) **Inverse:** every group operation g_i has an inverse (called g_i^{-1}) with the property $g_i \circ g_i^{-1} = e = g_i^{-1} \circ g_i$;

▶ The **topological properties** of the Lie group comes from the identification of each element in the group with a point in some topological space: $g_i \to g(x)$. In other words, the index i depends on one or more continuous real variables.

▶ The topological space that parameterizes the elements in a Lie group is a manifold. The Lie group is then seen as a smooth manifold, i.e. a differentiable manifold whose product and inverse operations are functions smooth on the manifold.

★ **Manifold:** Is a space that looks Euclidean on a small scale everywhere, but can have a very different large scale topology.

- For example

$$\text{Manifold } S^1 : \begin{cases} \text{Global: structure of a circle} \\ \\ \text{Local: structure of a straigth line, } \mathbb{R}^1 \end{cases}$$

▶ The topological notions of the Lie group allow us to introduce the ideas of compactness and noncompactness:

★ **Definition:** A topological space T is compact if every open cover (set of open sets U_α) has a finite subcover. Or, in other words, a space is compact if every infinite sequence of points has a subsequence that converges to a point in the space.

★ **Example:** the sphere S^2 is a compact space and the R^2 plane is not compact. The circle is compact and the hyperboloid is not compact.

▶ Compactness is an important topological property because it means that the space is in some sense like a bounded, closed space. For Lie groups it is important because all irreducible representations of a compact Lie group are finite dimensional and can be constructed by rather simple means (tensor products).

▶ We will not go into more details in the topological aspects of Lie groups because almost all of the Lie groups encountered in applications are matrix groups. This effects and enormous simplification in our study of Lie groups. Almost all of what we would like to learn about Lie groups can be determined by studying matrix groups.

▶ In terms of matrix groups we can define compactness as: A matrix Lie group G is said to be compact if the following two conditions are satisfied

(1) If A_m is any sequence of matrices in G, and A_m converges to a matrix A, then A is in G;

(2) There exists a constant C such that for all $A \in G$, $|A_{ij}| \leq C$ for all $1 \leq i, j \leq n$.

1.2 Matrix Lie Groups: some examples

- General and special linear groups

  The general linear group is denoted by $GL(n; \mathbb{F})$ and is our main example of a matrix group. Any of the other groups presented in these notes will be a subgroup of some $GL(n; \mathbb{F})$. We define it as

$$\boxed{\mathsf{GL}(n; \mathbb{F}) := \{g \text{ is } n \times n \text{ matrix with } \det(g) \neq 0\}} \tag{1.1}$$

The simplest subgroup of $GL(n;\mathbb{F})$ is the **special linear group** for the number filed $\mathbb{F}$ defined as

$$\boxed{\mathsf{SL}(n;\mathbb{F}) := \{g \in \mathsf{GL}(n;\mathbb{F}) |\, \det(g) = 1)\}} \tag{1.2}$$

Subgroups whose elements satisfy $\det(g) = 1$ are also called unimodular.

- (Pseudo) Orthogonal groups = (Indefinite) orthogonal groups

Choosing $\mathbb{F} = \mathbb{R}$, one has an important class of subgroups of $GL(n;\mathbb{R})$, the so called **(pseudo) orthogal groups**, defined by

$$\boxed{\mathsf{O}(p,q) := \left\{g \in \mathsf{GL}(n;\mathbb{R}) |\, gE^{(p,q)}g^T = E^{(p,q)}\right\}} \tag{1.3}$$

where $E^{(p,q)}$ is a $n \times n$ diagonal matrix with the first p entries $+1$ and the remaining q entries -1 (clearly $p + q = n$):

$$E^{(p,q)} := \mathrm{diag}(\underbrace{1,\cdots,1}_{p},\underbrace{-1,\cdots,-1}_{q})\,. \tag{1.4}$$

If either q or p are zero, the group is simply called orthogonal, otherwise pseudo-orthogonal. In this case one usually writes $\mathsf{O}(n)$ instead of $\mathsf{O}(n,0)$ or $\mathsf{O}(0,n)$. Taking the determinant of the defining relation, i.e. Eq. (1.3), lead us to

$$\det\left(gE^{(p,q)}g^T\right) = \det\left(E^{(p,q)}\right) \quad\Leftrightarrow\quad \det(g)^2\det\left(E^{(p,q)}\right) = \det\left(E^{(p,q)}\right)$$
$$\Rightarrow\quad \det(g)^2 = 1$$

and, therefore $\det(g) = \pm 1$ for $g \in \mathsf{O}(p,q)$.
Ex: The Lorentz group is $\mathsf{O}(1,3)$ or $\mathsf{O}(3,1)$ depending on metric convention.
Ex: Rotations and reflections in 3D space is $\mathsf{O}(3)$.

Those elements for which the determinant is $+1$ form a subgroup (of index two, two copies), and are called the **unimodular or special (pseudo) orthogonal groups**

$$\boxed{\mathsf{SO}(p,q) := \{g \in \mathsf{O}(p,q) |\, \det(g) = 1\}} \tag{1.5}$$

Here one usually writes $\mathsf{SO}(n)$ instead of $\mathsf{SO}(n,0)$ or $\mathsf{SO}(0,n)$.
Ex: Rotations in 3D space is $\mathsf{SO}(3)$.

- (Pseudo) Unitary groups = (Indefinite) Unitary groups

Next we look at the **(pseudo) unitary groups**, defined by

$$\boxed{\mathsf{U}(p,q) := \left\{g \in \mathsf{GL}(n;\mathbb{C}) |\, gE^{(p,q)}g^\dagger = E^{(p,q)}\right\}} \tag{1.6}$$

where † denotes the Hermitian conjugation. Here, the terminology is entirely analogous to the orthogonal groups, i.e. we simply speak of **unitary groups** is $p = 0$ or $q = 0$, in which case we write $U(n)$ instead of $U(n,0)$ or $U(0,n)$, otherwise of **pseudo unitary groups**. Taking the determinant of the above defining relation leads to

$$\det\left(gE^{(p,q)}g^{\dagger}\right) = \det\left(E^{(p,q)}\right) \quad \Leftrightarrow \quad |\det(g)|^2 \det\left(E^{(p,q)}\right) = \det\left(E^{(p,q)}\right)$$
$$|\det(g)|^2 = 1$$

and therefore $\det(g) = e^{i\theta}$ for $\theta \in [0, 2\pi[$. The subgroups of matrices with unit determinant are the **unimodular or special (pseudo) unitary groups**

$$\boxed{\mathsf{SU}(p,q) := \{g \in \mathsf{U}(p,q) | \det(g) = 1\}} \tag{1.7}$$

Again we write $\mathsf{SU}(n)$ instead of $\mathsf{SU}(n,0)$ or $\mathsf{SU}(0,n)$.
Ex: $\mathsf{SU}(2)$ of spin, $\mathsf{SU}_L(2)$ of electroweak force, $\mathsf{SU}(3)$ of color.

- Symplectic groups

  Let $\mathbb{I}_n$ be the unit $n \times n$ matrix, and $\hat{E}^{(2n)}$ the antisymmetric $2n \times 2n$ matrix

$$\hat{E}^{(2n)} := \begin{pmatrix} \mathbb{O}_n & \mathbb{I}_n \\ -\mathbb{I}_n & \mathbb{O}_n \end{pmatrix} \tag{1.8}$$

  We define $SP(2n;\mathbb{F})$, the **symplectic group** in $2n$ dimension over the field $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, by

$$\boxed{\mathsf{Sp}(2n;\mathbb{F}) := \left\{g \in \mathsf{GL}(2n;\mathbb{F}) | \, g\hat{E}^{(2n)}g^T = \hat{E}^{(2n)}\right\}} \tag{1.9}$$

1.3 Lie algebras: general concepts

▶ Two Lie groups are isomorphic if: their underlying manifolds are topologically equivalent; or the functions defining the group composition (multiplication) laws are equivalent.

▶ Showing the topological equivalence of two manifolds (can be smoothly deformed to each other, equal topological numbers) is not an easy task. Showing the equivalence of two composition laws is typically a much more difficult task (composition laws are in general nonlinear).

▶ The study of Lie groups would simplify greatly if the group composition law could somehow be linearized, and if this linearization retained a substantial part of the information of the original composition law. **Good news, this can be done!**

▶ A Lie group can be linearized in the neighborhood of any of its points. Linearization amounts to Taylor series expansion about the coordinates that define the group operation. What is being Taylor expanded is the group composition function.

▶ A Lie group is homogeneous: every point looks locally like every other point. This can be seen as follows:

- The neighborhood of group element a can be mapped into the neighborhood of group element b by multiplying a (and every element in its neighborhood) on the left by ba^{-1} (or on the right by $a^{-1}b$). This will map a into b and points near a to points near b

▶ It is therefore necessary to study the neighborhood of only one group operation in detail. A convenient point to choose is the identity.

▶ Linearization of a Lie group about the identity generates a new set of operators. These operators form a Lie algebra.

Lie algebras are constructed by linearizing Lie groups.

▶ Before defining a Lie algebra let us look into some concepts that come handy:

★ **Field:** A field $\mathbb{F}$ is a set of elements f_0, f_1, ... with:

- **Operation +**, called addition. $\mathbb{F}$ is an Abelian group under such operation, f_0 is the identity.
- **Operation ∘**, called scalar multiplication. Under such operation it shares many properties of a group: closure, associativity, existence of identity, existence of inverse except for f_0, distributive law ($f_i \circ (f_j + f_k) = f_i \circ f_j + f_i \circ f_k$).
- If $f_i \circ f_k = f_k \circ f_i$ the field is commutative;
- We will use $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$

★ **Linear vector space:** A linear vector space $\mathcal{V}$ consists of a collection of vectors $\vec{v}_1$, $\vec{v}_2$, ..., and a collection of f_1, f_2, ... $\in \mathbb{F}$, with two kinds of operations:

- **Vector addition +**, $(\mathcal{V}, +)$ is an Abelian group, $\vec{v}_1 + \vec{v}_2 = \vec{v}_2 + \vec{v}_1$
- **Scalar multiplication ∘**, sharing the following properties when $f_i \in \mathbb{F}$, $\vec{v}_j \in \mathcal{V}$

$f_i \circ \vec{v}_j \in \mathcal{V}$	Closure
$f_i \circ (f_j \circ \vec{v}_k) = (f_i \circ f_j) \circ \vec{v}_k$	Associativity
$1 \circ \vec{v}_i = \vec{v}_i = \vec{v}_i \circ 1$	Identity
$f_i \circ (\vec{v}_j + \vec{v}_k) = f_i \circ \vec{v}_j + f_i \circ \vec{v}_k$	Bilinearity

★ **Algebra:** A linear algebra consists of a collection of vectors $\vec{v}_1, \vec{v}_2, \ldots \in \mathcal{V}$ and a collection of $f_1, f_2, \ldots \in \mathbb{F}$, together with three kinds of operations:

- **Vector addition** $+$, satisfying the same postulates as in the linear vector space;
- **Scalar multiplication** $\circ$, satisfying the same postulates as in the linear vector space;
- **Vector multiplication** $\square$, with the following additional postulates for $\vec{v}_i \in \mathcal{V}$

$$\vec{v}_1 \square \vec{v}_2 \in \mathcal{V} \qquad \text{Closure}$$

$$(\vec{v}_1 + \vec{v}_2) \square \vec{v}_3 = \vec{v}_1 \square \vec{v}_3 + \vec{v}_1 \square \vec{v}_3 \qquad \text{Bilinearity}$$

Different varieties of algebras may be obtained, depending on which additional postulates are also satisfied: associativity, existence of identity, . . .

▶ **Definition of Lie algebra $\mathfrak{g}$:** It is an algebra, where vector multiplication has the following properties

a) The commutator of two elements is again an element of the algebra

$$a \square b \equiv [a, b] \in \mathfrak{g} \quad \forall a, b \in \mathfrak{g}\,.$$

b) A linear combination of elements of the algebra is again an element of the algebra

$$\alpha a + \beta b \in \mathfrak{g} \quad \text{if } a, b \in \mathfrak{g}.$$

Therefore the element 0 (zero) belongs to the algebra.

c) The following linearity is postulated

$$[\alpha a + \beta b, c] = \alpha[a, b] + \beta[b, c] \quad \text{for all } a, b, c \in \mathfrak{g}.$$

d) Interchanging both elements of a commutator result in the relation

$$[a, b] = -[b, a]\,.$$

e) Finally, the **Jacobi identity** has to be satisfied

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0\,.$$

Note that we do not demand that the commutators are associative, i.e. the relation $[a, [b, c]] = [[a, b], c]$ is not true in general.

f) In addition to the previous points we demand that a Lie algebra has a finite dimension n, i.e., it comprises a set of n linearly independent elements $e_1, \ldots, e_n$, which act as a basis, by which every element x of the algebra can be represented uniquely like

$$x = \sum_j^n \xi_j e_j \,.$$

In other words, the algebra constitutes an n-dimensional vector space (sometimes the dimension is named order). If the coefficients ξ_j and α, β are real, the algebra is named real. In a complex or complexified algebra the coefficients are complex.

A Lie algebra is a vector space with an alternate product satisfying the Jacobi identity.

▶ Due to condition *a*) the commutator of two basis elements belongs also to the algebra and therefore, following *f*), we get

$$[e_i, e_k] = \sum_{l=1}^n C_{ikl} e_l \,.$$

The n^3 coefficients C_{ijk} are called **structure constants** relative to the basis e_i. They are not invariant under a choice of basis.

▶Given a set of basis elements, the structure constants specify the Lie algebra completely. A Lie algebra with complex structure constants is complex itself.

▶ We can use the Jacobi identity to find a relation between the structure constants

$$\begin{aligned}
0 =& [e_i, [e_j, e_k]] + [e_j, [e_k, e_i]] + [e_k, [e_i, e_j]] \\
=& \sum_l C_{jkl}[e_i, e_l] + \sum_l [e_j, e_l] + \sum_l C_{ijl}[e_k, e_l] \\
=& \sum_{lm} C_{jkl} C_{ilm} e_m + \sum_{lm} C_{kil} C_{jlm} e_m + \sum_{lm} C_{ijl} C_{klm} e_m
\end{aligned}$$

Because the basis elements e_m are linearly independent, we get n equations for given values i, j, k

$$0 = \sum_l \left(C_{jkl} C_{ilm} + C_{kil} C_{jlm} + C_{ijl} C_{klm} \right) \,, \quad (m = 1, \ldots, n) \tag{1.10}$$

▶ There is also an antisymmetry relation in the first two indices of the structure constants, since

$$[e_i, e_k] = -[e_k, e_i] \quad \Leftrightarrow \quad \sum_l C_{ikl} e_l = -\sum_l C_{kil} e_l$$

Due to the linear independence of the e_i basis elements we get $C_{ikl} = -C_{kil}$. We will see that for $\mathfrak{su}(N)$ all indices are antisymmetric.

1.4 How good this really is?

▶ Linearization of a Lie group in the neighborhood of the identity to form a Lie algebra preserves the local properties but destroys the global ones, what happens far from the identity? Or in other words, can we recover the Lie group from its Lie algebra?

▶ Let us assume we have some operator X in a Lie algebra. Then if ϵ is a small real number, $I + \epsilon X$ represents an element in the Lie group close to the identity. We can attempt to move far from the identity by iterating this group operation many times

$$\lim_{k\to\infty} \left(I + \frac{1}{k}X\right)^k = \sum_{n=0}^{\infty} \frac{X^n}{n!} = \mathrm{EXP}(X) \quad \text{Exponential Map}$$

$$\text{Lie Group} = \mathrm{EXP}\,(\text{Lie Algebra})$$

▶ There are some very important questions to answer:

★ Does the exponential function map the Lie algebra back onto the entire Lie group?

★ Are the Lie groups with isomorphic Lie algebras themselves isomorphic?

▶ We shall explore these questions in the next sections.
▶ From the **Baker-Campbell-Hausdorff** formula:

$$e^a e^b = e^{a+b+\frac{1}{2}[a,b]+\frac{1}{12}[a,[a,b]]+\frac{1}{12}[a,[b,a]]+\cdots}$$

we see that if the commutator is in the algebra, the argument of the exponent on the right side is also in the algebra. The squared generators need not be in the algebra (often they are not) but the commutator must be.

Chapter 2

Let's rotate!

Rotational group in 2 and 3 dimensions

We look at some of the simplest and common groups in Physics: $SO(2)$ and $SO(3)$.
We look at the group definition, how to build the irreducible representations, ifinitesimal generators and consequently their Lie algebras

2.1 The rotational group in 2 dimensions

We define the abstract group of proper (no reflections) rotations $SO(2)$ to contain all rotations about the origin of a two-dimensional plane. The group as infinitely many elements, which can be specified using a continuous parameter $\alpha \in [0, 2\pi[$.

2.1.1 The $SO(2)$ group

The $SO(2)$ abstract group can be represented by $\mathsf{SO}(2)$ matrices

$$\boxed{\mathsf{SO}(2) := \left\{g \in \mathsf{GL}(2, \mathbb{R}) \,|\, gg^T = \mathbb{I},\, \det(g) = 1\right\}} \tag{2.1}$$

i.e. the set of unimodular, real and orthogonal 2×2 matrices.

$$\textsf{General structure: } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{with} \quad \det(A) = ad - bc = 1\,.$$

$$A \textsf{ is orthogonal: } \underbrace{\begin{pmatrix} a & c \\ b & d \end{pmatrix}}_{A^T} = \underbrace{\frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}_{A^{-1}} \quad \Rightarrow \quad d = a \quad \text{and} \quad c = -b\,.$$

$$\textsf{Orthogonality: } a^2 + b^2 = 1 \quad \Rightarrow \quad -1 \le \underbrace{a}_{\cos\alpha} \le 1 \quad -1 \le \underbrace{b}_{-\sin\alpha} \le 1\,.$$

▶ The general representation is then

$$\begin{aligned} R: \quad & SO(2) \to \mathsf{GL}(2,\mathbb{R}) \\ & R(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \quad \text{and} \quad \alpha \in [0, 2\pi[\,. \end{aligned} \tag{2.2}$$

Note: 2D rotations commute

$$R(\alpha_2)R(\alpha_1) = R(\alpha_1+\alpha_2) = R(\alpha_2+\alpha_1) = R(\alpha_1)R(\alpha_2). \tag{2.3}$$

i.e. the **group is Abelian**. Therefore, its complex irreducible representations are one-dimensional (ex: $e^{i\alpha}$). We then have that $SO(2)$ can be mapped to general real 2×2 matrices to form irreducible representations, or general complex 2×2 matrices, but in this case (as illustrated below) the representation is reducible:

$$\left[\begin{array}{llll} R: & SO(2) & \to \mathsf{GL}(2,\mathbb{R}) & \text{irreducible} \\ \\ D: & SO(2) & \to \mathsf{GL}(2,\mathbb{C}) & \text{reducible} \end{array}\right.$$

$SO(2)$ is an example of a **compact Lie group**, meaning roughly that it is a continuous group which can be parametrized by parameters in a finite interval.

2.1.2 The $SO(2)$ 1D irrep

▶ Following the definition of a representation, the 2×2 matrices $R(\alpha)$ might represent transformations $D(\alpha)$ in a complex two-dimensional vector space.

★ The eigenstate of $R(\alpha)$ are

$$\hat{u}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \quad \text{and} \quad \hat{u}_{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix} \tag{2.4}$$

with eigenvalues $\lambda_1(\alpha) = e^{-i\alpha}$ and $\lambda_{-1}(\alpha) = e^{i\alpha}$ as easily can be verified, ex:

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos\alpha - i\sin\alpha \\ \sin\alpha + i\cos\alpha \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\alpha} \\ ie^{-i\alpha} \end{pmatrix} \tag{2.5}$$

★ The similarity transformation (map of form $D^{(1)}(g) = SD^{(2)}(g)S^{-1}$) which transforms the matrix $R(\alpha)$ into $D(\alpha)$, is

$$\left[\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\right] R(\alpha) \left[\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\right] = \begin{pmatrix} e^{-i\alpha} & 0 \\ 0 & e^{i\alpha} \end{pmatrix} = D(\alpha), \tag{2.6}$$

$D(\alpha)$ is obviously reducible.

▶ The two resulting non-equivalent (cannot be related by similarity transformations) one-dimensional irreps of $SO(2)$ are then given by

$$D^{(1)}(\alpha) = e^{-i\alpha} \quad \text{and} \quad D^{(-1)}(\alpha) = e^{i\alpha} . \tag{2.7}$$

▶ We can then write

$$D(\alpha) = U(-\alpha) \oplus U(\alpha) \,, \quad \text{with} \quad U(\alpha) = e^{i\alpha} \tag{2.8}$$

Therefore we define the representation

$$U : \quad SO(2) \to \mathsf{GL}(1;\mathbb{C}) \quad \text{or} \quad \boxed{\mathsf{U}(1) := \left\{g \in \mathsf{GL}(1;\mathbb{C}) | \; gg^\dagger = 1)\right\}} \tag{2.9}$$

Since the representation U is bijective (one-to-one) on $\mathsf{U}(1)$,

$$\mathsf{SO}(2) \cong \mathsf{U}(1) \tag{2.10}$$

This can also be seen from the fact that U is injective (distinct elements in the domain are mapped to different elements in the image/codomain), using the first isomorphism theorem:

First Isomorphism Theorem

When given a homomorphism $f : G \to H$ (a group multiplication preserving map), we can identify two important subgroups:

★ The **image** of f, written as $\mathsf{im}\, f \subset H$. It is the set of all $h \in H$ which are mapped to by f

$$\mathsf{im}\, f := \{h \in H | \; h = f(g) \quad g \in G\}$$

★ The **kernel** of f, written as $\ker f \subset G$, is the set of all g that are mapped into the identity element $\mathbb{I}$ of H

$$\ker f := \{g \in G | \; f(g) = \mathbb{I}_H\} \,.$$

The theorem then reads: Let G and H be groups, and $f : G \to H$ be a homomorphism. We have

$$G/\ker f \cong \mathsf{im}\, f$$

Notice that the elements $U(\alpha) = e^{i\alpha}$ of $\mathsf{U}(1)$ lie along S^1, the unit circle in the complex plane

$$S^1 := \{z \in \mathbb{C}|\ |z| = 1\} \tag{2.11}$$

Thus

$$\boxed{SO(2) \cong \mathsf{SO}(2) \cong \mathsf{U}(1) \cong S^1 \cong [0, 2\pi[\cong \mathbb{R}/\mathbb{Z}}$$

▶ There are infinitely many non-equivalent irreps for $SO(2)$. We may indicate the various one-dimensional vector spaces by an index k in order to distinguish then, as well as the corresponding irrep, and the only basis vector of $\mathcal{V}^{(k)}$ by $\hat{u}_k$

$$\boxed{D^{(k)}(\alpha) = e^{-ik\alpha}, \quad k = 0, \pm 1, \pm 2, \cdots} \tag{2.12}$$

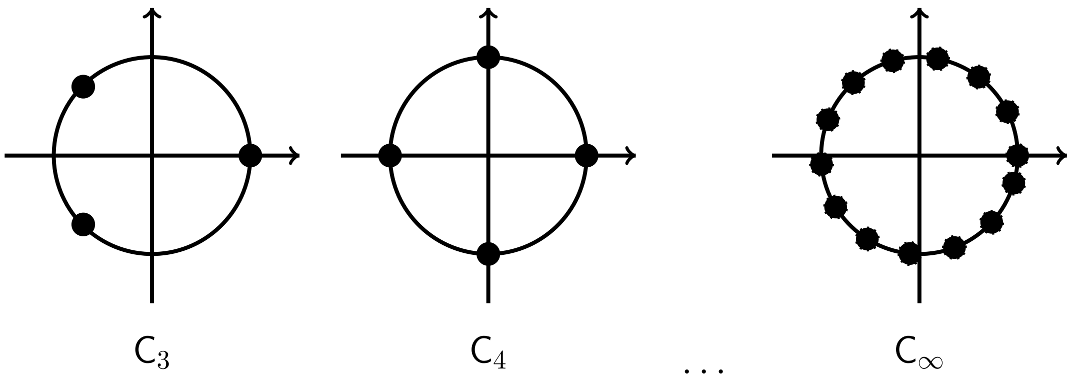
The representations $D^{(k)}$ are called the **standard irreps** of $SO(2)$.

▶ These are non-equivalent irreps because

$$\text{Similarity trans.: } s^{-1}\exp[-ik_1\alpha]s = \exp[-ik_2\alpha] \quad \text{if and only if } k_1 = k_2\,. \tag{2.13}$$

2.1.3 The infinitesimal generator of $SO(2)$

▶ Recall that C_n (the cyclic group with n elements) was generated by a single element a, i.e. rotation of $2\pi/n$. As we let n go to infinity, we notice that this rotation gets smaller and smaller. In this way a very small rotation $\varphi \to 0$ can be said to generate the group $SO(2)$.



★ Taylor expand the representation $R(\alpha)$ near the identity

$$R(\alpha) = R(0) + \alpha \underbrace{\frac{d}{d\alpha} R(\alpha)\Big|_{\alpha=0}}_{X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} + \cdots \tag{2.14}$$

The matrix X is called the **infinitesimal generator** of rotations in two dimensions.

★ For angles different from zero one has

$$\frac{d}{d\alpha}R(\alpha) = \begin{pmatrix} -\sin\alpha & -\cos\alpha \\ \cos\alpha & -\sin\alpha \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \\ = XR(\alpha)\,, \tag{2.15}$$

which is a differential equation, which has the solution

$$R(\alpha) = \exp\left[\alpha X\right] = \underbrace{\sum_{k=0}^{\infty} \frac{1}{k!}\alpha^k X^k}_{\text{def of exp(matrix)}} \;. \tag{2.16}$$

A representation $D(\alpha)$ for the group elements $R(\alpha)$ of $\mathsf{SO}(2)$, can be translated into a representation $d(X)$ of the infinitesimal generator X of the group, according to

$$\text{Exponential Map:} \quad \boxed{D(\alpha) = \exp\left[\alpha d(X)\right]\;.} \tag{2.17}$$

The operator X lives in the **tangent space** (the span of all tangent vectors) of $\mathsf{SO}(2)$ near the identity. X spans an **algebra**, a vector space endowed with a product with the property that two elements in the algebra can be "multiplied" and the result is still in the algebra.

► In the case of the standard irreps given in Eq. (2.12), we find for the representations $d^{(k)}(A)$ of the generator A of $SO(2)$, i.e.

$$d^{(k)}(A) = -ik\,, \quad k = 0, \pm1, \pm2, \cdots\;. \tag{2.18}$$

2.1.4 Representations of the Lie algebra $\mathfrak{so}(2)$

► We now turn our attention to a common representation used in quantum mechanics. Let $\mathcal{H}$ be the Hilbert space of quantum mechanical functions. We define the representation $D:\ \mathsf{G} \to \mathsf{GL}(\mathcal{H})$ by

$$D(R) \equiv \widehat{D}\,(\text{operator}) \qquad (\widehat{D}\psi)(\vec{r}) := \psi(R^{-1}\vec{r}) \quad R \in \mathsf{G} \tag{2.19}$$

This equation simply says that rotating a wave function in one direction is the same as rotating the coordinate axes in the other direction.This is a faithful representation
► We are interested in finding how the Lie algebra representations act on this space.
$D:\ \mathsf{SO}(2) \to \mathsf{GL}(\mathcal{H})$

★ We then have from the active rotation

$$\begin{aligned}
\text{rotation: } [\widehat{D}\psi](\vec{r}) &= \psi([R(\alpha)]^{-1}\vec{r})\,. \\
&\downarrow \\
\exp[\alpha\widehat{d}(A)]\psi(\vec{r}) &= \psi(\vec{r}+\Delta\vec{r}) \\
&\downarrow \\
\left[1+\alpha\widehat{d}(A)+\tfrac{\alpha^2}{2}[\widehat{d}(A)]^2+\cdots\right]\psi(\vec{r}) &= \psi(\vec{r})+\left(\Delta x\tfrac{\partial}{\partial x}+\Delta y\tfrac{\partial}{\partial y}\right)\psi(\vec{r})+ \\
&\quad +\tfrac{1}{2}\left((\Delta x)^2\tfrac{\partial^2}{\partial x^2}+2\Delta x\Delta y\tfrac{\partial^2}{\partial x\partial y}\right. \\
&\quad \left.+(\Delta y)^2\tfrac{\partial^2}{\partial y^2}\right)\psi(\vec{r})+\cdots
\end{aligned} \tag{2.20}$$

★ We need to expand Δx, Δy in α, i.e.

$$\Delta\vec{r}=\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix}=R(-\alpha)\vec{r}-\vec{r}=\begin{pmatrix}\alpha y-\frac{\alpha^2}{2}x+\cdots\\ -\alpha x-\frac{\alpha^2}{2}y+\cdots\end{pmatrix}\,. \tag{2.21}$$

★ Consequently, to first order in α

$$\left[1+\alpha\widehat{d}(A)\right]\psi(\vec{r})=\psi(\vec{r})+\alpha\left(y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}\right)\psi(\vec{r})\,. \tag{2.22}$$

★ Then we find for the **Hilbert space** functions $\psi(x,y)$, to first order in α, the following representation

$$d:\mathfrak{so}(2)\to\mathsf{GL}(\mathcal{H})\qquad \widehat{d}(A)=y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}\,. \tag{2.23}$$

► Now, one might like to inspect the higher order terms, the second order in α term is given by

$$\frac{\alpha^2}{2}\left(y^2\frac{\partial^2}{\partial x^2}-2xy\frac{\partial^2}{\partial x\partial y}+x^2\frac{\partial^2}{\partial y^2}\right)\psi(\vec{r})=\frac{\alpha^2}{2}\left(y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}\right)^2\psi(\vec{r})$$

So we find that also to second order in α, the solution is given by the same differential operator. In fact, this result is consistent with expansions up to any order in α.
► We can use **another parametrization**, the azimuthal parametrization

★ We introduce the azimuthal angle φ in the (x,y)-plane, according to

$$x=|\vec{r}|\cos\varphi\quad\text{and}\quad y=|\vec{r}|\sin\varphi\,, \tag{2.24}$$

then

$$[\widehat{D}\psi](|\vec{r}|,\varphi) = \psi(|\vec{r}|,\varphi-\alpha) = \psi(|\vec{r}|,\varphi)-\alpha\frac{\partial}{\partial\varphi}\psi(|\vec{r}|,\varphi)+\frac{\alpha^2}{2}\left(\frac{\partial}{\partial\varphi}\right)^2\psi(|\vec{r}|,\varphi)+\cdots \tag{2.25}$$

which lead us to the differential operator

$$\widehat{d}(A) = -\frac{\partial}{\partial\varphi}\,. \tag{2.26}$$

Because of this result, $\partial/\partial\varphi$ is sometimes referred to as the generator of rotations in the (x,y)-plane.

★ From Fourier-analysis we know that wave functions on the interval $[0,2\pi[$ has as a basis, the functions

$$\psi_m(\varphi) = e^{im\varphi}\,, \quad m = 0,\pm1,\pm2\cdots \tag{2.27}$$

Any **well-behaved** function $\psi(\varphi)$ ($\varphi \in [0,2\pi[$) can be expanded in a linear combination of the basis, i.e.

$$\psi(\varphi) = \sum_{m=-\infty}^{\infty} a_m\psi_m(\varphi)\,, \tag{2.28}$$

$$\text{Ortogonality:}\ (\psi_m,\psi_n) = \int_0^{2\pi}\frac{d\varphi}{2\pi}\psi_m^*(\varphi)\psi_n(\varphi) = \int_0^{2\pi}\frac{d\varphi}{2\pi}e^{i(n-m)\varphi} = \delta_{nm}\,.$$

$$\text{Coefficients:}\ a_m = (\psi_m,\psi) = \int_0^{2\pi}\frac{d\varphi}{2\pi}\psi_m^*(\varphi)\psi(\varphi) = \int_0^{2\pi}\frac{d\varphi}{2\pi}\psi(\varphi)e^{-im\varphi}\,. \tag{2.29}$$

► Physicists prefer x, $p = -i\partial/\partial x$ and $\bar{L} = -i\bar{r}\times\nabla$ ($\hbar = 1$). So, instead of the operator $d(A)$ of Eq. (2.23), we prefer

$$-ix\frac{\partial}{\partial y} + iy\frac{\partial}{\partial x} \quad \text{Angular Momentum operator}\,. \tag{2.30}$$

In order to suit that need, we take for the generator of $SO(2)$ the operator L, defined by

$$L = iA = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\,. \tag{2.31}$$

Physicists like Hermitian ($H^\dagger = H$) operators with real eigenvalues corresponding to observalbes.

The group elements $R(\alpha)$ of $SO(2)$ can then be written in the form

$$R(\alpha) = \exp[-i\alpha L]\,. \tag{2.32}$$

In this case the representation $d(L)$ of the generator L turns out to be

$$\widehat{d}(L) = -ix\frac{\partial}{\partial y} + iy\frac{\partial}{\partial x} = -i\frac{\partial}{\partial\varphi}\,. \tag{2.33}$$

★ Clearly, there exists no essential difference between the generator L and the generator A. It is just a matter of taste.

★ However, notice that whereas A is antisymmetric, L is hermitian. Both, A and L are traceless, since the group elements $R(\alpha)$ of $SO(2)$ are unimodular.

★ In the case of standard irreps, we find for the representation of $d^{(k)}(L)$ of the generator L the form

$$d^{(k)}(L) = k\,, \quad k = 0, \pm1, \pm2, \cdots\,. \tag{2.34}$$

▶ The special orthogonal group in two dimensions, $\mathsf{SO}(2)$, is generated by the operator L. This means that each element $R(\alpha)$ of the group can be written in the form $R(\alpha) = \exp[-i\alpha L]$. The reason that we only need one generator for the group $SO(2)$ is the fact that all group elements $R(\alpha)$ can be parameterized by one parameter α, representing the rotation angle.

▶ The operator L spans an **algebra**, which is a vector space endowed with a product. Since this algebra generates a Lie-group it's called a Lie-algebra

$$\mathfrak{so}(2) := \{X \in \mathbb{R}^{2\times2} \big|\ X^T + X = 0\}\,. \tag{2.35}$$

$$\begin{aligned}
R(\alpha)^T R(\alpha) = \mathbb{I} \quad &\Rightarrow & (\mathbb{I} + \alpha X^T + \cdots)(\mathbb{I} + \alpha X + \cdots) = \mathbb{I} \\
& & \alpha(X^T + X) = \mathbb{O} \\
\det(R(\alpha)) = 1 \quad &\Rightarrow & \det\left(e^{\alpha X}\right) = e^{\alpha \mathrm{Tr}(X)} = 1 \\
& & \mathrm{Tr}(X) = 0 \\
& & \text{Tr is already satisfied by the antisymmetric condition}
\end{aligned}$$

2.2 The rotational group in 3 dimensions

2.2.1 The rotation group $SO(3)$

▶ Just as $SO(2)$ contained all rotations around the origin of a two-dimensional plane, we define $SO(3)$ to be the abstract Lie group of all rotations about the origin of a three-dimensional Euclidean space $\mathbb{R}^3$

$$\mathsf{SO}(3) := \left\{ g \in \mathsf{GL}(3, \mathbb{R}) \middle| \, gg^T = \mathbb{I}, \, \det g = 1 \right\} \tag{2.36}$$

▶ An important difference compared to rotations in two dimensions is that in three dimensions rotations do not commute, i.e. 3D rotations form a non-Abelian group.

An arbitrary rotation can be characterized in various different ways:

★ Euler-Angle Parameterization

Any rotation in $\mathbb{R}^3$ can be described as a series of rotations about the $x-$, $y-$ and $z-$axis. The three rotations around the principal axes of the orthogonal coordinate system (x, y, z) are given by

$$\begin{aligned}
R(\hat{x}, \alpha) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}, \quad R(\hat{y}, \theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \\
R(\hat{z}, \varphi) &= \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned} \tag{2.37}$$

A combination of 3 rotations is always sufficient to reach any frame. The three elemental rotations may be **extrinsic** (rotations about the axes xyz of the original coordinate system, which is assumed to remain motionless), or **intrinsic** (rotations about the axes of the rotating coordinate system XYZ, solidary with the moving body, which changes its orientation after each elemental rotation).

Different authors may use different sets of rotation axes to define Euler angles, or different names for the same angles. Therefore, any discussion employing Euler angles should always be preceded by their definition.

Without considering the possibility of using two different conventions for the definition of the rotation axes (intrinsic or extrinsic), there exist twelve possible sequences of rotation axes, divided in two groups:

– **Proper Euler angles:** (**intrinsic**)
$(z-x-z, x-y-x, y-z-y, z-y-z, x-z-x, y-x-y)$
– **Tait-Bryan angles:** (**extrinsic**)
$(x-y-z, y-z-x, z-x-y, x-z-y, z-y-x, y-x-z)$

The parameterization known as **Euler parameterization** is given by the rotation matrix

$$R(\alpha,\beta,\gamma) = R(\hat{z},\alpha)R(\hat{x},\beta)R(\hat{z},\gamma)$$
$$= \begin{pmatrix} \cos\alpha\cos\gamma - \cos\beta\sin\alpha\sin\gamma & -\cos\alpha\sin\gamma - \cos\beta\cos\gamma\sin\alpha & \sin\alpha\sin\beta \\ \cos\gamma\sin\alpha + \cos\alpha\cos\beta\sin\gamma & \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & -\cos\alpha\sin\beta \\ \sin\beta\sin\gamma & \cos\gamma\sin\beta & \cos\beta \end{pmatrix}$$
$$\alpha,\, \gamma \in [0, 2\pi[\,, \quad \beta \in [0,\pi]\,.$$

★ Axis-Angle Parameterization

We can characterize a rotation by means of its rotation axis, which is the one-dimensional subspace of the three dimensional space which remains invariant under the rotation, and by its rotation angle. The direction of the rotation axis needs two parameters and the rotation angle gives the third.

2.2.2 The generators

▶ Using the first parameterization we can extract the **infinitesimal generators** of rotations in three dimensions, i.e.

$$A_1 = \frac{d}{d\alpha}R(\hat{x},\alpha)\Big|_{\alpha=0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad A_2 = \frac{d}{d\theta}R(\hat{y},\theta)\Big|_{\theta=0} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$
$$A_3 = \frac{d}{d\varphi}R(\hat{z},\varphi)\Big|_{\varphi=0} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.38}$$

▶ In terms of the Levi-Civita symbol, we can express the above matrix representation for the generators of $SO(3)$, by

$$(A_i)_{jk} = -\epsilon_{ijk}\,, \quad \epsilon_{ijk} = \begin{cases} +1 & \text{for} \quad ijk = 123, 312, 231\,. \\ -1 & \text{for} \quad ijk = 132, 213, 321\,. \\ 0 & \text{for all other combinations}\,. \end{cases} \tag{2.39}$$

A useful relation for the product of two Levi-Civita tensors is

$$\epsilon_{ijk}\epsilon_{ilm} = \epsilon_{1jk}\epsilon_{1lm} + \epsilon_{2jk}\epsilon_{2lm} + \epsilon_{3jk}\epsilon_{3lm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}\,. \tag{2.40}$$

Equipped with this knowledge let us compute the commutator of two generators. We have

$$\begin{aligned}
([A_i, A_j])_{kl} =& (A_iA_j)_{kl} - (A_jA_i)_{kl} = (A_i)_{km}(A_j)_{ml} - (A_j)_{km}(A_i)_{ml} \\
=& \epsilon_{ikm}\epsilon_{jml} - \epsilon_{jkm}\epsilon_{iml} = \epsilon_{mik}\epsilon_{mlj} - \epsilon_{mjk}\epsilon_{mli} \\
=& \delta_{il}\delta_{kj} - \delta_{ij}\delta_{kl} - (\delta_{jl}\delta_{ki} - \delta_{ji}\delta_{kl}) = \delta_{il}\delta_{kj} - \delta_{jl}\delta_{ki} \\
=& \epsilon_{mij}\epsilon_{mlk} = -\epsilon_{ijm}\epsilon_{mkl} = \epsilon_{ijm}(A_m)_{kl} \\
=& (\epsilon_{ijm}A_m)_{kl}\,.
\end{aligned} \tag{2.41}$$

▶ So, for the commutator of the generators we find

$$[A_i, A_j] = \epsilon_{ijm}A_m\,. \tag{2.42}$$

ϵ_{ijm} are called **structure constants**.

These infinitesimal generators define the Lie algebra

$$\boxed{\mathfrak{so}(3) := \{X \in \mathbb{R}^{3\times 3} \big|\ X^T + X = 0,\ \mathrm{Tr}(X) = 0\}\,.} \tag{2.43}$$

2.2.3 The Axis-Angle Parameterization

▶ In order to determine the Axis-Angle parameterization of a rotation in three dimensions, we define an arbitrary vector $\vec{n}$ by

$$\vec{n} = (n_1, n_2, n_3)\,, \tag{2.44}$$

as well as its **inner product** with the three generators, given by the expression

$$\vec{n}.\vec{A} = n_iA_i = n_1A_1 + n_2A_2 + n_3A_3\,. \tag{2.45}$$

▶ We then proceed as follows:

★ We compute higher order powers of this **inner product**. Actually, it is sufficient to determine the third power of the above expression, i.e. $(n^2 = \vec{n}.\vec{n})$

$$\begin{aligned}
\left[(\vec{n}.\vec{A})^3\right]_{ab} =& [(n_iA_i)(n_jA_j)(n_kA_k)]_{ab} = n_in_jn_k\,[A_iA_jA_k]_{ab} \\
=& n_in_jn_k(A_i)_{ac}(A_j)_{cd}(A_k)_{db} = -n_in_jn_k\epsilon_{iac}\epsilon_{jcd}\epsilon_{kdb} \\
=& -n_in_jn_k\,(\delta_{id}\delta_{aj} + \delta_{ij}\delta_{ad})\,\epsilon_{kdb} \\
=& -n_dn_an_k\epsilon_{kdb} + n^2n_k\epsilon_{kab} = 0 - n^2n_k(A_k)_{ab} \\
=& \left(-n^2\vec{n}.\vec{A}\right)_{ab}
\end{aligned} \tag{2.46}$$

★ Using the above relation repeatedly for the order powers of $\vec{n}.\vec{A}$, we may also determine its exponential, i.e.

$$\begin{aligned}\exp[\vec{n}.\vec{A}] =& \mathbb{I} + \vec{n}.\vec{A} + \frac{1}{2!}(\vec{n}.\vec{A})^2 + \frac{1}{3!}(\vec{n}.\vec{A})^3 + \frac{1}{4!}(\vec{n}.\vec{A})^4 + \cdot \\ =& \mathbb{I} + \vec{n}.\vec{A} + \frac{1}{2!}(\vec{n}.\vec{A})^2 + \frac{1}{3!}(-n^2\vec{n}.\vec{A}) + \frac{1}{4!}(-n^2(\vec{n}.\vec{A})^2) + \cdots \\ =& \mathbb{I} + \left(1 - \frac{n^2}{3!} + \frac{n^4}{5!} - \frac{n^6}{7!} + \cdots\right)(\vec{n}.\vec{A}) + \\ &+ \left(\frac{1}{2!} - \frac{n^2}{4!} + \frac{n^4}{6!} - \frac{n^6}{8!} + \cdots\right)(\vec{n}.\vec{A})^2 \\ =& \mathbb{I} + \left(n - \frac{n^3}{3!} + \frac{n^5}{5!} - \frac{n^7}{7!} + \cdots\right)(\hat{n}.\vec{A}) + \qquad (\text{note } \hat{n}) \\ &+ \left(\frac{n^2}{2!} - \frac{n^4}{4!} + \frac{n^6}{6!} - \frac{n^8}{8!} + \cdots\right)(\hat{n}.\vec{A})^2\end{aligned} \tag{2.47}$$

We then get

$$\exp[\vec{n}.\vec{A}] = \mathbb{I} + \sin n(\hat{n}.\vec{A}) + (1 - \cos n)(\hat{n}.\vec{A})^2 \,. \tag{2.48}$$

★ The exponential operator leaves the vector $\vec{n}$ invariant, i.e.

$$\exp[\vec{n}.\vec{A}]\vec{n} = [\mathbb{I} + \vec{n}.\vec{A} + \cdots]\vec{n} = \vec{n}\,. \tag{2.49}$$

This is easy to see this by looking at

$$[(\vec{n}.\vec{A})\vec{n}]_i = (\vec{n}.\vec{A})_{ij}n_j = (n_kA_k)_{ij}n_j = n_k(A_k)_{ij}n_j = -n_k\epsilon_{kij}n_j = 0\,, \tag{2.50}$$

or equivalently

$$(\vec{n}.\vec{A})\vec{n} = 0 \tag{2.51}$$

★ The axis through the vector $\vec{n}$ is invariant, which implies that it is **the rotation axis**, as expected.

▶ The matrix $\vec{n}.\vec{A}$ takes the explicit form

$$\vec{n}.\vec{A} = \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix} \tag{2.52}$$

which clearly is a traceless and antisymmetric matrix.

▶ We have then found a second parameterization of a rotation around the origin in three dimensions

Rotation parameterization:	$R(\varphi,\hat{n}) = \exp[\varphi\hat{n}.\vec{A}]$
Rotation angle:	$\varphi \equiv n = \sqrt{n_1^2+n_2^2+n_3^2}$
Rotation axis:	$\hat{n} = \vec{n}/n$

$$R(\varphi,\hat{n}) = \exp\left[\varphi\hat{n}.\vec{A}\right] = \mathbb{I}_3 + \sin\varphi\left(\hat{n}.\vec{A}\right) + (1-\cos\varphi)\left(\hat{n}.\vec{A}\right)^2$$
$$= \begin{pmatrix} \cos\varphi + \hat{n}_1^2(1-\cos\varphi) & \hat{n}_1\hat{n}_2(1-\cos\varphi) - \hat{n}_3\sin\varphi & \hat{n}_1\hat{n}_3(1-\cos\varphi) + \hat{n}_2\sin\varphi \\ \hat{n}_1\hat{n}_2(1-\cos\varphi) + \hat{n}_3\sin\varphi & \cos\varphi + \hat{n}_2^2(1-\cos\varphi) & \hat{n}_2\hat{n}_3(1-\cos\varphi) - \hat{n}_1\sin\varphi \\ \hat{n}_1\hat{n}_3(1-\cos\varphi) - \hat{n}_2\sin\varphi & \hat{n}_2\hat{n}_3(1-\cos\varphi) + \hat{n}_1\sin\varphi & \cos\varphi + \hat{n}_3^2(1-\cos\varphi) \end{pmatrix}$$

▶ This parameterization is very useful because it makes use of the infinitesimal generators of $\mathsf{SO}(3)$.

2.2.4 The $\mathfrak{so}(3)$ Lie-algebra

▶ Instead of the antisymmetric generators A_i ($i = 1,2,3$), we prefer to continue with the Hermitian generators $L_i = iA_i$, i.e.

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.53}$$

▶ An arbitrary rotation $R(\hat{n},\alpha)$ with a rotation angle α around the axis spanned by $\hat{n}$ can be expressed as

$$R(\hat{n},\alpha) = \exp[-i\alpha\hat{n}.\vec{L}] = \mathbb{I} + \sin\alpha(-i\hat{n}.\vec{L}) + (1-\cos\alpha)(-i\hat{n}.\vec{L})^2\,. \tag{2.54}$$

★ The Lie-algebra is spanned by A_i or $-iL_i$.

★ The product of the Lie-algebra is given by the **Lie-product**, defined by the commutator of two elements of the algebra

$$[L_i, L_j] = i\epsilon_{ijm}L_m\,. \tag{2.55}$$

This establishes their relation with the angular momentum in Quantum Mechanics.

★ Since the $SO(3)$ generator space is an algebra, any real linear combination of $-iL_i$'s can serve as a generator of a rotation.

★ We may extend the algebra to include complex linear combinations. While those operators do not, in general, represent rotations, they can be very useful for the construction of the representations of $SO(3)$. (We shall explore this in detail when working with the $SU(2)$ group)

★ Two such combinations are (the ladder operators)

$$L_{\pm} = L_1 \pm iL_2 \,, \tag{2.56}$$

satisfying

$$[L_3, L_{\pm}] = \pm L_{\pm} \quad \text{and} \quad [L_+, L_-] = 2L_3 \tag{2.57}$$

with $L_{\pm}^{\dagger} = L_{\mp}$.

2.2.5 The Casimir operator

▶ The **first lemma of Schur** states that a matrix which commutes with all matrices $D(g)$ of an irreducible representation D must be proportional to the unit matrix.

Let us find the matrix that commutes will all rotations:

★ We represent such a matrix by the generator combination X, i.e. $\exp[-iX]$, satisfying

$$\exp[-iX]\exp[-i\alpha\hat{n}.\vec{L}] = \exp[-i\alpha\hat{n}.\vec{L}]\exp[-iX] \,. \tag{2.58}$$

★ Only true when X commutes with $\hat{n}.\vec{L}$. The Baker-Campbell-Hausdorff formula:

$$e^a e^b = e^{a+b+\frac{1}{2}[a,b]+\frac{1}{12}[a,[a,b]]+\frac{1}{12}[a,[b,a]]+\cdots}$$

★ Consequently, X must be an operator which commutes with all three generators L_i. A possible solution is

$$X = \vec{L}^2 = L_1^2 + L_2^2 + L_3^2 \tag{2.59}$$

which satisfies

$$[\vec{L}^2, L_i] = 0 \quad \text{for} \quad i = 1, 2, 3 \,. \tag{2.60}$$

★ The operator X (or L^2) is called the (second order) **Casimir** operator of $SO(3)$.

★ L^2 is not an element of the Lie-algebra, because it can not be written as a linear combination of generators. It is important for the classification of the irreps.

★ We can also express it in term of the ladder operators

$$
\begin{aligned}
L^2 =& L_3^2 + L_1^2 + L_2^2 = L_3^2 + \frac{1}{4}(L_+ + L_-)^2 - \frac{1}{4}(L_+ - L_-)^2 \\
=& L_3^2 + \frac{1}{2}(L_+L_- + L_-L_+) \\
=& L_3^2 + L_3 + L_-L_+ \\
=& L_3^2 - L_3 + L_+L_- \qquad (2.61)
\end{aligned}
$$

2.2.6 The $(2\ell+1)$-dimensional irrep d^ℓ

A representation of the Lie-algebra and hence of the Lie-group itself, is fully characterized once the transformations of $d^{(\ell)}(L_i)$ (for $i = 1, 2, 3$) of a certain complex vector space $\mathcal{V}_\ell$ onto itself are known.

We take the following strategy:

★ Select an orthonormal basis in $\mathcal{V}_\ell$ such that $d^{(\ell)}(L_3)$ is diagonal in that basis. (convenience)

★ From $SO(2)$ we know that the eigenvalues of the rotations around the z-axis represented by the generator L_3 are integer numbers.

★ We might label the basis vectors of $\mathcal{V}_\ell$ according to the eigenvalues for $d^{(\ell)}(L_3)$, i.e.

$$
d^{(\ell)}(L_3)\,|\ell, m\rangle = m\,|\ell, m\rangle \quad \text{for} \quad m = 0, \pm 1, \pm 2, \cdots \qquad (2.62)
$$

The Casimir operator L^2

$$
L^2 = [d^{(\ell)}(L_1)]^2 + [d^{(\ell)}(L_2)]^2 + [d^{(\ell)}(L_3)]^2 \qquad (2.63)
$$

commutes with $[d^{(\ell)}(L_i)]$. Thus, from Schur's first lemma

$$
L^2\,|\ell, m\rangle = \underbrace{\lambda}_{=\ell(\ell+1)}\,|\ell, m\rangle \quad \text{for all basis vectors.} \qquad (2.64)
$$

I will show during the lecture that $\lambda = \ell(\ell+1)$.

► The non-negative integer ℓ characterizes the irrep $d^{(\ell)}$ of $SO(3)$. To each possible values corresponds a non-equivalent unitary irrep. This way we obtain for $\mathcal{V}_\ell$ the following basis (see lecture)

$$
|\ell, -\ell\rangle\ ,\ |\ell, -\ell+1\rangle\ ,\cdots,\ |\ell, \ell-1\rangle\ ,\ |\ell, \ell\rangle\ . \qquad (2.65)
$$

Its dimension is determined by the structure of the basis, i.e.

$$
\dim(d^{(\ell)}) = 2\ell + 1\,. \qquad (2.66)
$$

▶ At this basis L_3 is represented by a diagonal $(2\ell+1)\times(2\ell+1)$ matrix of the form

$$d^{(\ell)}(L_3) = \begin{pmatrix} \ell & & & & \\ & \ell-1 & & & \\ & & \cdots & & \\ & & & -\ell+1 & \\ & & & & -\ell \end{pmatrix} \quad \text{traceless and Hermitain.} \tag{2.67}$$

We can now see how the raising/lowering operators act on the vector basis:

★ Using the commutation relations for the generators (2.57) we get

$$d^{(\ell)}(L_3)d^{(\ell)}(L_\pm) = d^{(\ell)}(L_\pm L_3 + [L_3, L_\pm]) = d^{(\ell)}(L_\pm)d^{(\ell)}(L_3) \pm d^{(\ell)}(L_\pm)\,.$$

★ Acting this operator in the vector basis we get

$$\begin{aligned} d^{(\ell)}(L_3)\left(d^{(\ell)}(L_\pm)\,|\ell,m\rangle\right) =& d^{(\ell)}(L_\pm)\left(d^{(\ell)}(L_3) \pm 1\right)|\ell,m\rangle \\ =&(m\pm 1)\left(d^{(\ell)}(L_\pm)\,|\ell,m\rangle\right) \end{aligned}$$

★ We can then conclude that

$$d^{(\ell)}(L\pm)\,|\ell,m\rangle = C_\pm(\ell,m)\,|\ell,m\pm 1\rangle\ .$$

with $C_\pm(\ell,m)$ some constant of proportionality

★ The constant can be determined by

$$\begin{aligned} |C_\pm(\ell,m)|^2 =& \left|d^{(\ell)}(L_\pm)\,|\ell,m\rangle\right|^2 \\ =& \langle\ell,m|\,[d^{(\ell)}(L_\pm)]^\dagger d^{(\ell)}(L_\pm)\,|\ell,m\rangle \\ =& \langle\ell,m|\left\{L^2 - \left([d^{(\ell)}(L_3)]^2 \pm d^{(\ell)}(L_3)\right)\right\}|\ell,m\rangle \\ =&\ell(\ell+1) - m(m\pm 1) \end{aligned} \tag{2.68}$$

★ It is convenient not to consider a phase factor for the coefficient $C_\pm(\ell,m)$ and to take the following real solutions

$$\boxed{C_\pm(\ell,m) = \sqrt{\ell(\ell+1) - m(m\pm 1)}\,.} \tag{2.69}$$

▶ Let us take as an example the three dimensional irrep $d^{(1)}$ of the $\mathfrak{so}(3)$ Lie algebra

★ At the basis of eigenstates of $d^{(1)}(L_3)$ given by

$$|1,1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |1,0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |1,-1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{2.70}$$

Therefore

$$d^{(1)}(L_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{2.71}$$

★ For the raising and lowering operators we get

$$\begin{cases} d^{(1)}(L_+)\,|1,-1\rangle = \sqrt{2}\,|1,0\rangle \\ d^{(1)}(L_+)\,|1,0\rangle = \sqrt{2}\,|1,1\rangle \\ d^{(1)}(L_+)\,|1,1\rangle = 0 \end{cases}, \quad \begin{cases} d^{(1)}(L_-)\,|1,-1\rangle = 0 \\ d^{(1)}(L_-)\,|1,0\rangle = \sqrt{2}\,|1,-1\rangle \\ d^{(1)}(L_-)\,|1,1\rangle = \sqrt{2}\,|1,0\rangle \end{cases} \tag{2.72}$$

in matrix form

$$d^{(1)}(L_+) = \sqrt{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad d^{(1)}(L_-) = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.73}$$

★ The matrices $d^{(1)}(L_{1,2})$ can be obtained from the above expressions, i.e.

$$d^{(1)}(L_1) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad d^{(1)}(L_2) = \frac{1}{i\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \tag{2.74}$$

★ We can now study the related representation $D^{(1)}$ of the group elements of $SO(3)$

$$D^{(1)}(\hat{n},\alpha) = \exp[-i\alpha\hat{n}.d^{(1)}(\vec{L})]\,. \tag{2.75}$$

★ Rotation around the x-axis

$$\begin{aligned}D^{(1)}(R(\hat{x},\alpha)) =&\exp[-i\alpha d^{(1)}(L_1)]\\ =&\mathbb{I}+\sin\alpha(-id^{(1)}(L_1))+(1-\cos\alpha)(-id^{(1)}(L_1))^2\\ =&\begin{pmatrix}\frac{1}{2}(\cos\alpha+1) & -\frac{i}{\sqrt{2}}\sin\alpha & \frac{1}{2}(\cos\alpha-1)\\ -\frac{i}{\sqrt{2}}\sin\alpha & \cos\alpha & -\frac{i}{\sqrt{2}}\sin\alpha\\ \frac{1}{2}(\cos\alpha-1) & -\frac{i}{\sqrt{2}}\sin\alpha & \frac{1}{2}(\cos\alpha+1)\end{pmatrix}\end{aligned} \tag{2.76}$$

★ Rotation around the y-axis

$$\begin{aligned}D^{(1)}(R(\hat{y},\theta)) =&\exp[-i\alpha d^{(1)}(L_2)]\\ =&\mathbb{I}+\sin\theta(-id^{(1)}(L_2))+(1-\cos\theta)(-id^{(1)}(L_2))^2\\ =&\begin{pmatrix}\frac{1}{2}(1+\cos\theta) & -\frac{1}{\sqrt{2}}\sin\theta & \frac{1}{2}(1-\cos\theta)\\ \frac{1}{\sqrt{2}}\sin\theta & \cos\theta & -\frac{1}{\sqrt{2}}\sin\theta\\ \frac{1}{2}(1-\cos\theta) & \frac{1}{\sqrt{2}}\sin\theta & \frac{1}{2}(1+\cos\theta)\end{pmatrix}\end{aligned} \tag{2.77}$$

★ Rotation around the z-axis

$$\begin{aligned}D^{(1)}(R(\hat{z},\varphi)) =&\exp[-i\varphi d^{(1)}(L_3)]\\ =&\mathbb{I}+\sin\varphi(-id^{(1)}(L_3))+(1-\cos\varphi)(-id^{(1)}(L_3))^2\\ =&\begin{pmatrix}e^{-i\varphi} & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & e^{i\varphi}\end{pmatrix}\end{aligned} \quad . \tag{2.78}$$

Note that these rotation matrices are not the same as the ones presented in the beginning of this section.

2.2.7 Standard irreps in terms of spherical harmonics

▶ $SO(3)$ is the symmetry group rotations in 3D

▶ **Spherical harmonics** serve as a well-behaved function basis

$$Y_{\ell m}(\theta,\varphi)\,, \quad \text{for } \ell=0,1,3,\cdots \quad \text{and } m=-\ell,\cdots,\ell \tag{2.79}$$

Some properties of the wave functions:

★ The innerproduct for two functions χ and ψ is defined as

$$(\psi, \chi) = \int_{\text{sphere}} d\Omega\, \psi^*(\theta, \varphi)\chi(\theta, \varphi)\,. \tag{2.80}$$

★ The spherical harmonics form an orthogonal basis for this innerproduct

$$\int_{\text{sphere}} d\Omega\, Y^*_{\lambda\mu}(\theta, \varphi) Y_{\ell m}(\theta, \varphi) = \delta_{\lambda\ell}\delta_{\mu m}\,. \tag{2.81}$$

★ Each function $f(\theta, \varphi)$ on the sphere can be expanded in terms of the spherical harmonics

$$f(\theta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} B_{\ell m} Y_{\ell m}(\theta, \varphi)\,. \tag{2.82}$$

★ The coefficients are given by

$$B_{\ell m} = \int_{\text{sphere}} d\Omega Y^*_{\ell m}(\theta, \varphi) f(\theta, \varphi) \tag{2.83}$$

► An active rotation $R(\hat{n}, \alpha)$ in three dimensions induces a transformation in function space given by (unit sphere)

$$\widehat{D}(\hat{n}, \alpha) f(\vec{r}) = f([R(\hat{n}, \alpha)]^{-1}\vec{r}) \quad \text{with} \quad \vec{r}(\theta, \varphi) = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} \tag{2.84}$$

Let us study the differential operators that follow from the three generators L_i:

★ Rotation around the z-axis $R(\hat{z}, \alpha)$ in leading order in α

$$\begin{aligned} \vec{r}(\theta + \Delta\theta, \varphi + \Delta\varphi) =& R(\hat{z}, \alpha)^{-1}\vec{r}(\theta, \varphi) = \begin{pmatrix} 1 & \alpha & 0 \\ -\alpha & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \vec{r}(\theta, \varphi) \\ =& \begin{pmatrix} \sin\theta(\cos\varphi + \alpha\sin\varphi) \\ \sin\theta(-\alpha\cos\varphi + \sin\varphi) \\ \cos\theta \end{pmatrix} \end{aligned} \tag{2.85}$$

Since

$$\vec{r}(\theta + \Delta\theta, \varphi + \Delta\varphi) \simeq \vec{r}(\theta, \varphi) + \Delta\theta \frac{\partial}{\partial\theta}\vec{r}(\theta, \varphi) + \Delta\varphi \frac{\partial}{\partial\varphi}\vec{r}(\theta, \varphi) \tag{2.86}$$

we find

$$\Delta\theta = 0 \quad \text{and} \quad \Delta\varphi = -\alpha\,. \tag{2.87}$$

We then get

$$\begin{aligned} \widehat{D}(\hat{z},\alpha) f(\vec{r}) &= f([R(\hat{z},\alpha)]^{-1}\vec{r}) \\ \Leftrightarrow \left(1 - i\alpha\widehat{d}(L_3)\right) f(\vec{r}) &= \left(1 - \alpha\frac{\partial}{\partial\varphi}\right) f(\vec{r}) \end{aligned}$$

Form which we get

$$\boxed{\widehat{d}(L_3) = -i\frac{\partial}{\partial\varphi}\,.} \tag{2.88}$$

★ Rotation around the x-axis $R(\hat{x},\alpha)$ in leading order in α

$$\vec{r}(\theta+\Delta\theta,\varphi+\Delta\varphi) = \begin{pmatrix} 1 & 0 & \\ 0 & 1 & \alpha \\ 0 & -\alpha & 1 \end{pmatrix} \vec{r}(\theta,\varphi) = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi + \alpha\cos\theta \\ -\alpha\sin\theta\sin\varphi + \cos\theta \end{pmatrix} \tag{2.89}$$

we then have

$$\Delta\theta = \alpha\sin\varphi \quad \text{and} \quad \Delta\varphi = \alpha\cot\theta\cos\varphi \tag{2.90}$$

which ultimately leads to the identification

$$\boxed{\widehat{d}(L_1) = i\left(\sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right)} \tag{2.91}$$

★ Rotation around the y-axis $R(\hat{y},\alpha)$ in leading order in α

$$\vec{r}(\theta+\Delta\theta,\varphi+\Delta\varphi) = \begin{pmatrix} 1 & 0 & -\alpha \\ 0 & 1 & 0 \\ \alpha & 0 & 1 \end{pmatrix} \vec{r}(\theta,\varphi) = \begin{pmatrix} \sin\theta\cos\varphi - \alpha\cos\theta \\ \sin\theta\sin\varphi \\ \alpha\sin\theta\cos\varphi + \cos\theta \end{pmatrix} \tag{2.92}$$

we then have

$$\Delta\theta = -\alpha\cos\varphi \quad \text{and} \quad \Delta\varphi = \alpha\cot\theta\sin\varphi \tag{2.93}$$

which ultimately leads to the identification

$$\boxed{\widehat{d}(L_2) = i\left(-\cos\varphi\frac{\partial}{\partial\theta} + \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right)} \tag{2.94}$$

▶ The raising and lowering operators $L_\pm$ are then given by

$$\widehat{d}(L_\pm) = e^{\pm i\varphi}\left(i\cot\theta\frac{\partial}{\partial\varphi} \pm \frac{\partial}{\partial\theta}\right) \tag{2.95}$$

▶ For the Casimir operator L^2 we get

$$\begin{aligned}
L^2 =&[\widehat{d}(L_1)]^2 + [\widehat{d}(L_2)]^2 + [\widehat{d}(L_3)]^2 \\
=& -\left[\sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right]^2 - \left[-\cos\varphi\frac{\partial}{\partial\theta} + \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right]^2 - \frac{\partial^2}{\partial\varphi^2} \\
=& -\left[\sin\varphi\frac{\partial^2}{\partial\theta^2} - \frac{\sin\varphi\cos\varphi}{\sin^2\theta}\frac{\partial}{\partial\varphi} + \cot\theta\sin\varphi\cos\varphi\frac{\partial^2}{\partial\theta\partial\varphi} + \cot\theta\cos^2\varphi\frac{\partial}{\partial\theta}\right. \\
&\left. + \cot\theta\cos\varphi\sin\varphi\frac{\partial^2}{\partial\varphi\partial\theta} - \cot^2\theta\cos\varphi\sin\varphi\frac{\partial}{\partial\varphi} + \cot^2\theta\cos^2\varphi\frac{\partial^2}{\partial\varphi^2}\right] \\
&- \left[\cos\varphi\frac{\partial^2}{\partial\theta^2} + \frac{\sin\varphi\cos\varphi}{\sin^2\theta}\frac{\partial}{\partial\varphi} - \cot\theta\sin\varphi\cos\varphi\frac{\partial^2}{\partial\theta\partial\varphi} + \cot\theta\sin^2\varphi\frac{\partial}{\partial\theta}\right. \\
&\left. - \cot\theta\cos\varphi\sin\varphi\frac{\partial^2}{\partial\varphi\partial\theta} + \cot^2\theta\cos\varphi\sin\varphi\frac{\partial}{\partial\varphi} + \cot^2\theta\sin^2\varphi\frac{\partial^2}{\partial\varphi^2}\right] - \frac{\partial^2}{\partial\varphi^2} \\
=& -\left[\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right] \\
=& -\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]
\end{aligned} \tag{2.96}$$

The spherical harmonics $Y_{\ell m}(\theta,\varphi)$ form a standard basis of a $(2\ell+1)$-dimensional irrep in function space

★ For L_3 operator we get the following differential equation

$$-i\frac{\partial}{\partial\varphi}Y_{\ell m}(\theta,\varphi) = \widehat{d}(L_3)Y_{\ell m}(\theta,\varphi) = mY_{\ell m}(\theta,\varphi) \tag{2.97}$$

★ For the Casimir operator L^2 we get

$$\begin{aligned}
-\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]Y_{\ell m}(\theta,\varphi) =& L^2 Y_{\ell m}(\theta,\varphi) \\
=& \ell(\ell+1)Y_{\ell m}(\theta,\varphi)\,.
\end{aligned} \tag{2.98}$$

▶ Let us look into the solutions of the above differential equations:

★ The first equation is simple and has the following solution

$$Y_{\ell m}(\theta,\varphi) = X_{\ell m}(\theta)e^{im\varphi} \tag{2.99}$$

where $X_{\ell m}(\theta)$ is some yet unknown function.

★ For $m = 0$ the second differential equation reads

$$-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}X_{\ell 0}(\theta) = \ell(\ell+1)X_{\ell 0}(\theta) \tag{2.100}$$

Change variables

$$\xi = \cos\theta \quad \text{and} \quad P_\ell(\xi) = X_{\ell 0}(\theta) \tag{2.101}$$

leading to the equation

$$\textbf{Legendre's diff. eq.:} \quad \boxed{-\frac{d}{d\xi}(1-\xi^2)\frac{d}{d\xi}P_\ell(\xi) = \ell(\ell+1)P_\ell(\xi)} \tag{2.102}$$

$$
\begin{aligned}
&P_0(\xi) = 1\\
&P_1(\xi) = \xi\\
&P_2(\xi) = \tfrac{1}{2}(3\xi^2 - 1)\\
&P_3(\xi) = \tfrac{1}{2}(5\xi^3 - 3\xi)\\
&P_4(\xi) = \tfrac{1}{8}(35\xi^4 - 30\xi^2 + 3)\\
&P_5(\xi) = \tfrac{1}{8}(63\xi^5 - 70\xi^3 + 15\xi)\\
&\ldots
\end{aligned}
$$

▶ For the spherical harmonics with $m \neq 0$ we can use the raising and lowering operators.

Let us construct the complete basis of spherical harmonics for the standard irrep corresponding to $\ell = 1$:

- For $m = 0$ we have

$$Y_{1,0}(\theta,\varphi) = N\cos\theta\,, \tag{2.103}$$

with the normalization constant determined by the orthogonality relation, i.e.

$$1 = |N|^2\int_0^{2\pi} d\varphi\int_{-1}^{+1} d\cos\theta\,\cos^2\theta = 4\pi|N|^2/3 \tag{2.104}$$

leading to the conventional choice $N = \sqrt{3/4\pi}$. Therefore,

$$\boxed{Y_{1,0}(\theta,\varphi) = \sqrt{\frac{3}{4\pi}}\cos\theta\,.} \tag{2.105}$$

- For $m = +1$ we get

$$\begin{aligned}\sqrt{2}Y_{1,+1}(\theta,\varphi) =& \widehat{d}(L_+)Y_{1,0}(\theta,\varphi) = e^{i\varphi}\left(i\cot\theta\frac{\partial}{\partial\varphi} + \frac{\partial}{\partial\theta}\right)\sqrt{\frac{3}{4\pi}}\cos\theta \\ =& -e^{i\varphi}\sqrt{\frac{3}{4\pi}}\sin\theta\end{aligned} \tag{2.106}$$

leading to

$$\boxed{Y_{1,+1}(\theta,\varphi) = -\sqrt{\frac{3}{8\pi}}e^{i\varphi}\sin\theta} \tag{2.107}$$

- For $m = -1$ we get

$$\begin{aligned}\sqrt{2}Y_{1,-1}(\theta,\varphi) =& \widehat{d}(L_-)Y_{1,0}(\theta,\varphi) = e^{-i\varphi}\left(i\cot\theta\frac{\partial}{\partial\varphi} - \frac{\partial}{\partial\theta}\right)\sqrt{\frac{3}{4\pi}}\cos\theta \\ =& \sqrt{\frac{3}{4\pi}}e^{-i\varphi}\sin\theta\end{aligned} \tag{2.108}$$

leading to the solution

$$\boxed{Y_{1,-1}(\theta,\varphi) = \sqrt{\frac{3}{8\pi}}e^{-i\varphi}\sin\theta} \tag{2.109}$$

Chapter 3

$2\pi \neq 4\pi$!

Unitary group in 2 dimensions

We look at one of the most common groups in Physics: $SU(2)$. We look at the group definition, Lie algebra, its relation with the group of rotations in $3D$. We build the irreducible representations and study the product space of irreps in $SU(2)$

3.1 The $SU(2)$ group

▶ The special unitary group $SU(2)$ is defined as

$$\boxed{\mathsf{SU}(2) := \left\{ g \in \mathsf{GL}(2;\mathbb{C}) |\, gg^\dagger = \mathbb{I}_2 \,,\, \det(g) = 1 \right\}} \tag{3.1}$$

The most general form of a unitary matrix in two dimensions is given by

$$U(a,b) = \begin{pmatrix} a^* & -b^* \\ b & a \end{pmatrix}\,, \quad a, b \in \mathbb{C}\,. \tag{3.2}$$

and $|a|^2 + |b|^2 = 1$.

A convenient parametrization of a unitary 2×2 matrices is by means of the **Cayley-Klein** parameters $\xi_{0,1,2,3}$.

★ These parameters are related with a and b, i.e.

$$a = \xi_0 + i\xi_3 \quad \text{and} \quad b = \xi_2 - i\xi_1\,. \tag{3.3}$$

★ The Unitary matrix can then be written as

$$U(\xi_0, \vec{\xi}) = \xi_0 \mathbb{I} - i\vec{\xi}.\vec{\sigma}\,, \tag{3.4}$$

with σ_i the **Pauli** matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{3.5}$$

★ Since $U(\xi_0, \vec{\xi})$ is unimodular we have

$$\det(U) = (\xi_0)^2 + (\xi_1)^2 + (\xi_2)^2 + (\xi_3)^2 = 1 . \tag{3.6}$$

Therefore, we only have 3 free parameters to describe a 2×2 unimodular unitary matrix.

★ Combine the 3 parameters n_i into a vector $\vec{n}$

$$\vec{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} . \tag{3.7}$$

★ One might select the Cayley-Klein parameters as

$$\xi_0 = \cos\frac{\varphi}{2} \quad \text{and} \quad \vec{\xi} = \hat{n}\sin\frac{\varphi}{2} \tag{3.8}$$

with

$$\hat{n} = \frac{1}{n}\vec{n} \quad \text{and} \quad \varphi \equiv n = \sqrt{(n_1)^2 + (n_2)^2 + (n_3)^2} . \tag{3.9}$$

★ We then find the **axis-angle parametrization**

$$\begin{aligned} U(\hat{n}, \varphi) =& \mathbb{I}\cos\frac{\varphi}{2} - i(\hat{n}.\vec{\sigma})\sin\frac{\varphi}{2} \\ =& \begin{pmatrix} \cos\frac{\varphi}{2} - i\hat{n}_3\sin\frac{\varphi}{2} & -(\hat{n}_2 + i\hat{n}_1)\sin\frac{\varphi}{2} \\ (\hat{n}_2 - i\hat{n}_1)\sin\frac{\varphi}{2} & \cos\frac{\varphi}{2} + i\hat{n}_3\sin\frac{\varphi}{2} \end{pmatrix} \end{aligned}$$

The Pauli matrices follow the following relations

- The product

$$\sigma_i\sigma_j = \mathbb{I}\delta_{ij} + i\epsilon_{ijk}\sigma_k . \tag{3.10}$$

- The commutator

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \tag{3.11}$$

as seen from the product

- The anti-commutator

$$\{\sigma_i, \sigma_j\} = \sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}\mathbb{I} . \tag{3.12}$$

as seen from the product

3.2 The $\mathfrak{su}(2)$ algebra

▶ The group of special unitary transformations in two dimensions, i.e. $SU(2)$, is a Lie group. This can be easily understood if one considers the exponential representation, i.e.

$$\exp[\alpha^i X_i] \in \mathsf{SU}(2) \quad \text{if} \quad X_i \in \mathfrak{su}(2)\,. \tag{3.13}$$

then

$$\exp[\alpha^i X_i]^\dagger \exp[\alpha^i X_i] = (\mathbb{I} + \alpha^i X_i^\dagger + \cdots)(\mathbb{I} + \alpha^i X_i + \cdots) = \mathbb{I} \quad \Rightarrow X_i^\dagger + X_i = 0$$

Therefore

(3.14)

▶ A basis for $\mathfrak{su}(2)$ is given by the matrices $s_j = -\frac{i}{2}\sigma_j$

$$s_1 = -\frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad s_2 = -\frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad s_3 = -\frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$
$$\mathfrak{su}(2) = \mathrm{span}_{\mathbb{R}}\{s_1, s_2, s_3\}$$

elements of the $\mathfrak{su}(2)$ algebra are real linear combinations of s_i.

▶ In a similar ways to what we have done for the $\mathsf{SO}(3)$ group, we can parameterize the group in terms of 3 rotations. The **Euler parametrization** is then given by

$$\begin{aligned} U(\alpha, \beta, \gamma) =& \exp[\alpha s_3]\exp[\beta s_2]\exp[\gamma s_3] \\ =& \begin{pmatrix} e^{-\frac{i}{2}(\alpha+\gamma)} \cos\frac{\beta}{2} & -e^{\frac{i}{2}(\alpha-\gamma)} \sin\frac{\beta}{2} \\ e^{-\frac{i}{2}(\alpha-\gamma)} \sin\frac{\beta}{2} & e^{\frac{i}{2}(\alpha+\gamma)} \cos\frac{\beta}{2} \end{pmatrix} \\ & \alpha \in [0, 2\pi[\,, \quad \beta \in [0, \pi]\,, \quad \gamma \in [0, 4\pi[\end{aligned}$$

Instead of using the anti-hermitian matrices s_i, we can use space of traceless hermitian matrices

$$\boxed{H_2 := \left\{X \in \mathbb{C}^{2\times 2} \,|\, X^\dagger - X = 0\,,\ \mathrm{Tr}(X) = 0\right\} = \mathrm{span}_{\mathbb{R}}\{\sigma_1, \sigma_2, \sigma_2\}} \tag{3.15}$$

Therefore, for this choice we

★ The generators of $\mathfrak{su}(2)$ to be

$$J_1 = \frac{\sigma_1}{2}\,, \quad J_2 = \frac{\sigma_2}{2} \quad \text{and} \quad J_3 = \frac{\sigma_3}{2} \tag{3.16}$$

★ The arbitrary special unitary transformation $U(\vec{n} = \alpha\hat{n})$ in two dimensions may be written as

$$U(\hat{n}, \alpha) = e^{-i\alpha\hat{n}.\vec{J}} \tag{3.17}$$

★ The commutation relation is

$$[J_i, J_j] = i\epsilon_{ijk} J_k \tag{3.18}$$

★ The commutation relation of the $\mathsf{SU}(2)$ generators $\vec{J}$ are identical to the commutation relations of the $\mathsf{SO}(3)$ generators $\vec{L}$. Both generators are traceless and hermitian.

▶ We can regard H_2, the space of hermitian 2×2 matrices, as a real three-dimensional vector space by considering the isomorphism

$$\begin{aligned} f:\, \mathbb{R}^3 &\to H_2 \\ f(\hat{e}_i) &= \sigma_i \end{aligned} \tag{3.19}$$

In this way, we can think of any 2×2 hermitian matrix $\mathcal{X}$ as a real vector $\vec{x}$

$$\vec{x} = x^i \hat{e}_i \quad \longrightarrow \quad \mathcal{X} = f(\vec{x}) = x^i \sigma_i \tag{3.20}$$

▶ On this linear space we define the scalar product to be

$$\mathcal{X}.\mathcal{Z} := \frac{1}{2}\mathrm{Tr}(\mathcal{X}\mathcal{Z})$$

3.3 Relation between $SU(2)$ and $SO(3)$

Theorem:

To rotate a vector $\mathcal{X}$ in H_2, we can use a 2×2 unitary matrix $U \in \mathsf{SU}(2)$ in the following way

$$\mathcal{X} \to U\mathcal{X}U^\dagger$$

We prove this theorem by considering the scalar product

$$\begin{aligned} \mathcal{X}.\mathcal{Z} :=& \frac{1}{2}\mathrm{Tr}(\mathcal{X}\mathcal{Z}) \\ =& \frac{1}{2}\mathrm{Tr}\left(\sigma_i \sigma_j\right) x^i z^j = \frac{1}{2}(2\delta_{ij}) x^i z^j = x^i z^i \end{aligned}$$

which defines the concept of an angle on H_2. And also show that

★ $U\mathcal{X}U^\dagger$ is still an element of H_2

$$\begin{aligned} \text{hermitian:}\ & (U\mathcal{X}U^\dagger)^\dagger = U\mathcal{X}U^\dagger\,, \\ \text{Traceless:}\ & \mathrm{Tr}(U\mathcal{X}U^\dagger) = \mathrm{Tr}(\mathcal{X}U^\dagger U) = \mathrm{Tr}(\mathcal{X}) = 0 \end{aligned}$$

★ The mapping $\mathcal{X} \to U\mathcal{X}U^\dagger$ is isometric (conserves length)

$$(U\mathcal{X}U^\dagger).(U\mathcal{Z}U^\dagger) = \frac{1}{2}\mathrm{Tr}(U\mathcal{X}U^\dagger U\mathcal{Z}U^\dagger) = \frac{1}{2}\mathrm{Tr}(\mathcal{X}\mathcal{Z}) = \mathcal{X}.\mathcal{Z}$$

▶ A transformation $U(\hat{n},\alpha) \in \mathsf{SU}(2)$ is given by

$$\begin{aligned} U(\hat{n},\alpha) =& \exp\left[-\frac{i}{2}\alpha\hat{n}^k\sigma_k\right] = \mathbb{I}\cos\frac{\alpha}{2} - i\hat{n}^k\sigma_k\sin\frac{\alpha}{2} \\ =& \begin{pmatrix} \cos\frac{\alpha}{2} - i\hat{n}^3\sin\frac{\alpha}{2} & -(i\hat{n}^1+\hat{n}^2)\sin\frac{\alpha}{2} \\ -(i\hat{n}^1-\hat{n}^2)\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} + i\hat{n}^3\sin\frac{\alpha}{2} \end{pmatrix} \end{aligned} \tag{3.21}$$

▶ Let us now consider a rotation α around the z-axis of H_2 (i.e. $\hat{n}^3 = 1$, $\hat{n}_1 = \hat{n}_2 = 0$, the σ_3-axis)

$$\begin{aligned} U\mathcal{X}U^\dagger \equiv& U(\hat{z},\alpha)\mathcal{X}U(\hat{z},\alpha)^\dagger = U(\hat{z},\alpha)\mathcal{X}U(\hat{z},-\alpha) \\ =& \begin{pmatrix} e^{-\frac{i\alpha}{2}} & 0 \\ 0 & e^{\frac{i\alpha}{2}} \end{pmatrix} (x\sigma_1 + y\sigma_2 + z\sigma_3) \begin{pmatrix} e^{\frac{i\alpha}{2}} & 0 \\ 0 & e^{-\frac{i\alpha}{2}} \end{pmatrix} \\ =& x\begin{pmatrix} 0 & e^{-i\alpha} \\ e^{i\alpha} & 0 \end{pmatrix} + iy\begin{pmatrix} 0 & -e^{-i\alpha} \\ e^{i\alpha} & 0 \end{pmatrix} + z\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ =& (x\cos\alpha - y\sin\alpha)\sigma_1 + (x\sin\alpha + y\cos\alpha)\sigma_2 + z\sigma_3 \end{aligned}$$

▶ We see that rotating $\mathcal{X} \in H_2$ around the σ_3-axis using $U \in \mathsf{SU}(2)$

$$\mathcal{X} \to U\mathcal{X}U^\dagger \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} x\cos\alpha - y\sin\alpha \\ x\sin\alpha + y\cos\alpha \\ z \end{pmatrix} \tag{3.22}$$

is identical to rotating $\vec{x} \in \mathbb{R}^3$ around the $\hat{e}_3$-axis using $R \in \mathrm{SO}(3)$

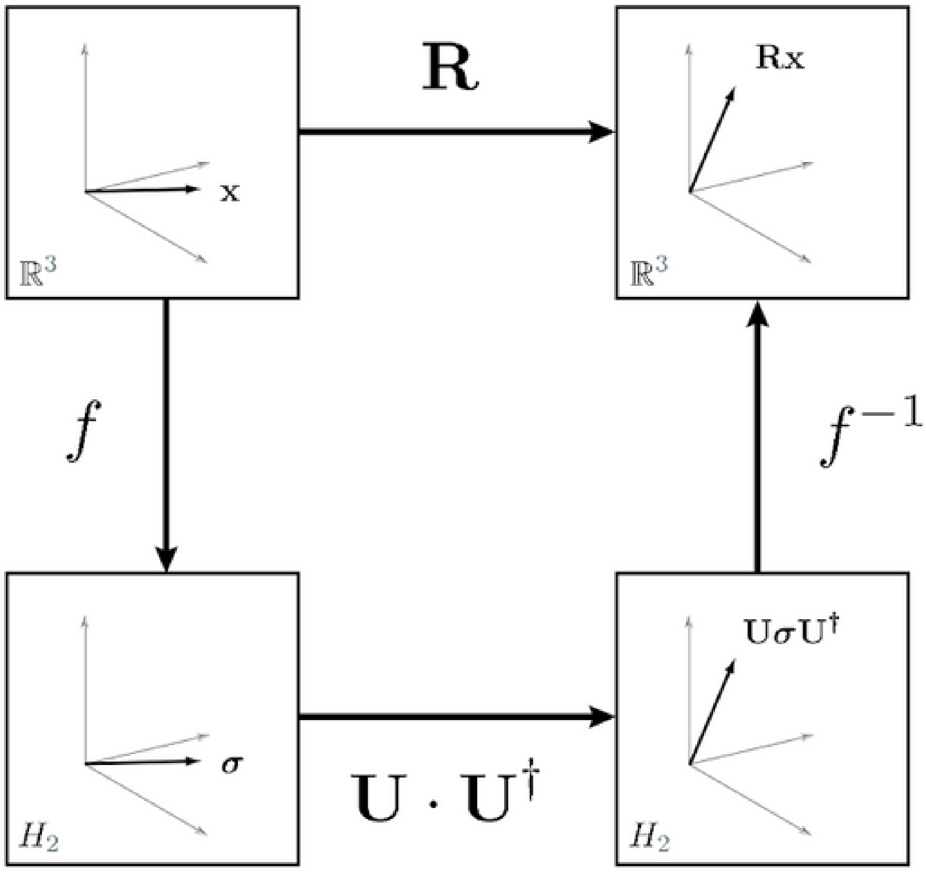
$$\vec{x} \to R\vec{x} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} x\cos\alpha - y\sin\alpha \\ x\sin\alpha + y\cos\alpha \\ z \end{pmatrix} \tag{3.23}$$

▶ We see that the groups $\mathsf{SU}(2)$ and $\mathsf{SO}(3)$, which may not seem similar at first sight, are actually quite intimately related.

We express this relationship concretely in the form of a homomorphism

$$\begin{aligned}\Phi : \mathsf{SU}(2) &\to \mathsf{SO}(3)\\ \Phi(U)\vec{x} &:= f^{-1}(Uf(\vec{x})U^\dagger)\\ &= R\vec{x}\end{aligned}$$

▶ This can be easily visualize in the figure below



Theorem:

We have for the homormophism $\Phi : \mathsf{SU}(2) \to \mathsf{SO}(3)$ the following properties:

1) Φ is surjective (if every possible image is mapped to by at least one argument)

2) $\ker \Phi = \{\mathbb{I}, -\mathbb{I}\} =: Z_2$

▶ Because of 2), we see that Φ is not injective. In fact

$$\Phi(-U)\vec{x} = f^{-1}(-Uf(\vec{x})(-U)^\dagger) = f^{-1}(Uf(\vec{x})U^\dagger) = \Phi(U)\vec{x} = R\vec{x}$$

That is, every element $R \in \mathsf{SO}(3)$ is mapped by two elements of $\mathsf{SU}(2)$

$$\Phi^{-1}(R) = \{U, -U\}\,. \quad \text{(two-fold covering)} \tag{3.24}$$

▶ Form the first isomorphism theorem for algebras we get

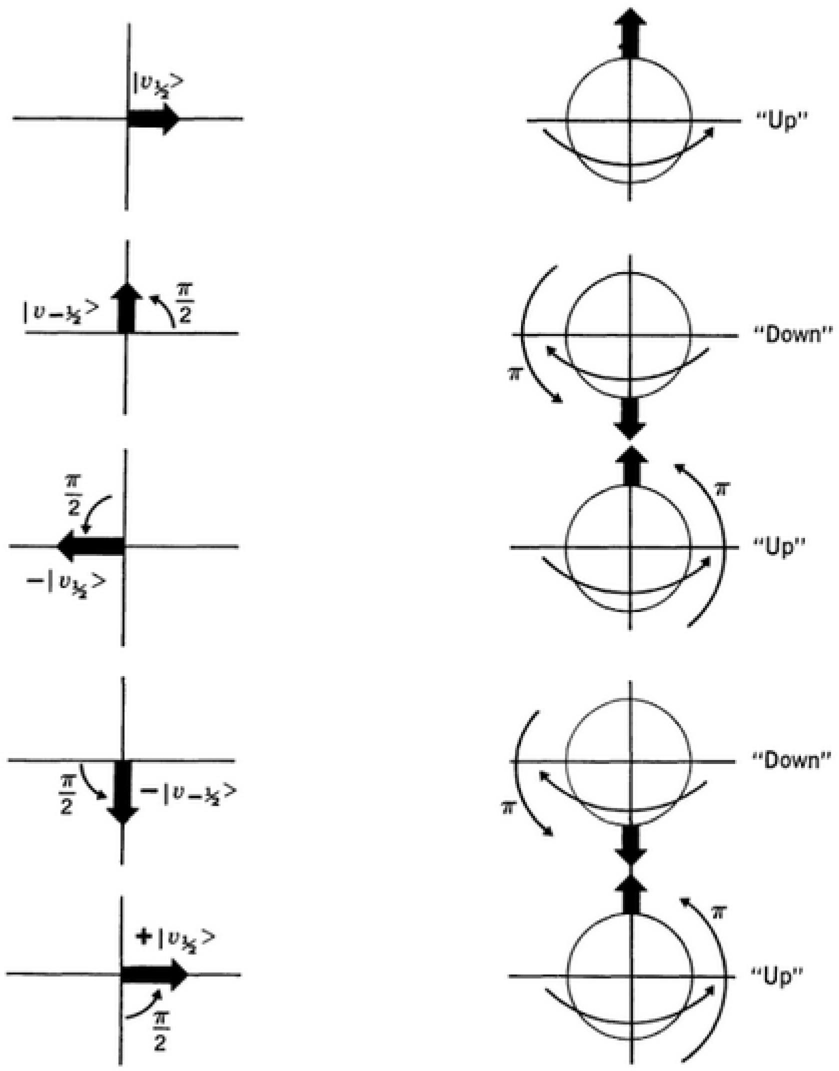
$$\mathsf{SU}(2)/Z_2 \cong \mathsf{SO}(3) \tag{3.25}$$

▶ Although $\mathsf{SU}(2)$ and $\mathsf{SO}(3)$ are not globally isomorphic, they are locally isomorphic. $\mathsf{SU}(2)$ is the **universal covering group** of $\mathsf{SO}(3)$, or the **double-cover**.

▶ Comparing the two rotation matrices $R(\varphi, \hat{n})$ for $\mathsf{SO}(3)$ and $U(\varphi, \hat{n})$ for $\mathsf{SU}(2)$ we see that

$$R(2\pi, \hat{n}) = \mathbb{I}_3 \quad \text{while} \quad U(2\pi, \hat{n}) = -\mathbb{I}$$

Only after 4π does $U(\varphi, \hat{n})$ returns to the same point, during this time $R(\varphi, \hat{n})$ has covered the points twice.



3.4 The subgroup $U(1)$

▶ The matrices $U(\hat{z},\alpha)$ form a subgroup $U(1)$ of $SU(2)$, because of the following property

$$U(\alpha_2,\hat{z})U(\alpha_1,\hat{z}) = U(\alpha_1+\alpha_2,\hat{z})\,, \quad U(\alpha,\hat{z}) = \begin{pmatrix} e^{-\frac{i}{2}\alpha} & 0 \\ 0 & e^{\frac{i}{2}\alpha} \end{pmatrix} \tag{3.26}$$

At the subspace spanned by the vector $\hat{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $U(\alpha,\hat{z})$ is represented by

$$U(\alpha,\hat{z})\hat{e}_1 = e^{-\frac{i}{2}\alpha}\hat{e}_1 \quad \text{so} \quad D(U(\hat{z},\alpha)) = e^{-\frac{i}{2}\alpha} \tag{3.27}$$

▶ Other 1D representations may be given by

$$D^{(m)}(\alpha, U(\hat{z})) = e^{-im\alpha} \tag{3.28}$$

Since only 4π angles bring us back to the unit we have

$$U(4\pi,\hat{z}) = \mathbb{I} \quad \Rightarrow \quad e^{-im4\pi} = 1\,. \tag{3.29}$$

which can be solved for the values

$$m = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 2, \cdots\,. \tag{3.30}$$

3.5 The $(2j+1)$-dimensional irrep

▶ Since $SU(2)$ generators $\vec{J}$ and $SO(3)$ generators $\vec{L}$ satisfy the same commutation relations, the structure of the Lie-algebras for the two groups is the same.

We can repeat the construction of irreps followed for $SO(3)$. However, there is one important difference:

- In $SO(3)$, because m is an integer, only odd values $(2\ell+1)$ for the irrep dimension are possible.
- For $SU(2)$, because j can be half-integer, also even $(2j+1)$ dimensional irreps are possible.

★ The irreps of $SU(2)$ are characterized by the parameter j

$$j = 0,\ \frac{1}{2},\ 1,\ \frac{3}{2},\ 2,\ \cdots \tag{3.31}$$

★ The $(2j+1)$-dimensional irrep of the Lie-algebra of $SU(2)$, $d^{(j)}$, is characterized by the $(2j+1)$ orthogonal basis vectors

$$|j,j\rangle\ , \quad |j,j-1\rangle\,,\cdots,|j,-j+1\rangle\,,\ |j,-j\rangle \tag{3.32}$$

★ The basis is chosen such that J_3 is represented by the diagonal, i.e.

$$d^{(j)}(J_3)\,|j,m\rangle = m\,|j,m\rangle\ . \tag{3.33}$$

★ The raising and lowering operators $J_\pm$ as defined as

$$J_\pm = J_1 \pm iJ_2\,. \tag{3.34}$$

★ For the $(2j+1)$-dimensional irrep $d^{(j)}$ they are represented by

$$d^{(j)}(J_\pm)\,|j,m\rangle = \sqrt{j(j+1)-m(m\pm 1)}\,|j,m\pm 1\rangle\ . \tag{3.35}$$

★ The Casimir operator, J^2, is defined by

$$J^2 = \left(d^{(j)}(J_1)\right)^2 + \left(d^{(j)}(J_2)\right)^2 + \left(d^{(j)}(J_3)\right)^2 \tag{3.36}$$

with eigenvalues

$$J^2\,|j,m\rangle = j(j+1)\,|j,m\rangle\ . \tag{3.37}$$

Besides allowing for more possible irreps for $SU(2)$, the representations have the same structure as the representations of $SO(3)$.

▶ Therefore the general representation for the J_i are

$$\begin{aligned}
d^{(j)}(J_1)^{m'}{}_m =& \frac{1}{2}\left(\sqrt{j(j+1)-m(m+1)}\delta^{m'}_{m+1} + \sqrt{j(j+1)-m(m-1)}\delta^{m'}_{m-1}\right)\\
d^{(j)}(J_2)^{m'}{}_m =& \frac{1}{2i}\left(\sqrt{j(j+1)-m(m+1)}\delta^{m'}_{m+1} - \sqrt{j(j+1)-m(m-1)}\delta^{m'}_{m-1}\right)\\
d^{(j)}(J_3)^{m'}{}_m =& m\delta^{m'}_m
\end{aligned}$$

▶ **The two-dimensional irrep**

- For $j = 1/2$ one has the two-dimensional irrep $d^{(1/2)}$ of the Lie-algebra of $SU(2)$.
- We denote the basis vectors by

$$\left|\frac{1}{2},\frac{1}{2}\right\rangle = \begin{pmatrix}1\\0\end{pmatrix} \quad \text{and} \quad \left|\frac{1}{2},-\frac{1}{2}\right\rangle = \begin{pmatrix}0\\1\end{pmatrix} \tag{3.38}$$

- The generator J_3 is in this basis represented by a diagonal matrix. Its eigenvalues are indicated in the notation used for the basis vectors, i.e. $\pm 1/2$. Thus

$$d^{(1/2)}(J_3) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\sigma_3}{2} \tag{3.39}$$

- For the raising and lowering operators we get

$$\begin{aligned} d^{(1/2)}(J_+)\left|\frac{1}{2},\frac{1}{2}\right\rangle &= 0 \\ d^{(1/2)}(J_+)\left|\frac{1}{2},-\frac{1}{2}\right\rangle &= \left|\frac{1}{2},\frac{1}{2}\right\rangle \\ d^{(1/2)}(J_-)\left|\frac{1}{2},+\frac{1}{2}\right\rangle &= \left|\frac{1}{2},-\frac{1}{2}\right\rangle \\ d^{(1/2)}(J_-)\left|\frac{1}{2},-\frac{1}{2}\right\rangle &= 0 \end{aligned} \tag{3.40}$$

  Leading to the following matrix representations (note that we get this from the above equation)

$$d^{(1/2)}(J_+) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}(\sigma_1 + i\sigma_2) \quad \text{and} \quad d^{(1/2)}(J_-) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \frac{1}{2}(\sigma_1 - i\sigma_2) \tag{3.41}$$

- This leads to the matrix representation

$$d^{(1/2)}(J_1) = \frac{\sigma_1}{2} \quad \text{and} \quad d^{(1/2)}(J_2) = \frac{\sigma_2}{2}\,. \tag{3.42}$$

- The Casimir operator is $J^2 = \frac{3}{4}\mathbb{I}$, which agrees with

$$J^2\left|\frac{1}{2},m\right\rangle = \frac{1}{2}\left(\frac{1}{2}+1\right)\left|\frac{1}{2},m\right\rangle = \frac{3}{4}\left|\frac{1}{2},m\right\rangle \tag{3.43}$$

- The matrices of the representation $d^{(1/2)}$ are exactly the same as the matrices of the definition of the Lie-algebra of $SU(2)$.

► **The three-dimensional irrep.**

- For $j = 1$ one has the three-dimensional irrep $d^{(1)}$ of $SU(2)$

- The basis vectors are given by

$$|1,1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} , \quad |1,0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} , \quad |1,-1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{3.44}$$

- The generator J_3 is represented by

$$d^{(1)}(J_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{3.45}$$

  s.t. $d^{(1)}(J_3)\,|\ell, m\rangle = m\,|\ell, m\rangle$

- The raising and lowering operators have the matrix representation

$$d^{(1)}(J_+) = \sqrt{2} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad d^{(1)}(J_-) = \sqrt{2} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{3.46}$$

  Note: these matrices are exactly equal to the corresponding matrices used in $SO(3)$.

- The representation of the group elements $U(\hat{n}, \alpha)$ of $SU(2)$ is given by

$$D^{(1)}(U(\hat{n}, \alpha)) = e^{-i\alpha \hat{n}.d^{(1)}(\vec{J})} \tag{3.47}$$

- For example for $U(\hat{z}, \alpha)$ we have

$$D^{(1)}(U(\hat{z}, \alpha)) = \begin{pmatrix} e^{-i\alpha} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\alpha} \end{pmatrix} \tag{3.48}$$

  The group elements $U(\hat{z}, \alpha)$ and $U(\hat{z}, \alpha + 2\pi)$ from (3.21) are not equal, but their representations are. The representation D^1 is therefore not faithful. This is true for all odd dimensions.

- The reason is that there are always two different group elements $U(\hat{n}, \alpha)$ and $U(\hat{n}, \alpha + 2\pi)$ which corresponds to one rotation $R(\hat{n}, \alpha)$ in $\mathbb{R}^3$ and that the odd dimensional irreps are equivalent for $SO(3)$ and $SU(2)$.

- The Casimir operator is

$$J^2 = (J_3)^2 + J_3 + J_- J_+ = 2\mathbb{I}\,. \tag{3.49}$$

3.6 A short note on $\mathfrak{sl}(2,\mathbb{C})$

One small but vital detail that is often left out in many physics-oriented texts is the consideration of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$:

$$\mathfrak{sl}(2,\mathbb{C}) := \{X \in \mathbb{C}^{2\times 2} \big| \ \mathrm{Tr}(X) = 0\} \tag{3.50}$$

which is the Lie algebra of the Lie group $\mathsf{SL}(2,\mathbb{C})$.

▶ This algebra is naturally a vector space over the complex numbers as traceless matrices remain traceless by multiplication by a complex number. The algebra is therefore a complex Lie algebra.

▶ We can choose as a basis of generators

$$J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad J_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \tilde{J}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\tilde{J}_3$ is not exactly the same as J_3 defined before, there is a overall 1/2 factor difference. The Lie product gives

$$[J_+, J_-] = J_3\,, \quad [J_3, J_\pm] = \pm 2 J_\pm\,.$$

▶ Even though the algebra is complex we can easily get a real algebra since the commutator of the algebra elements have only real numbers.

▶ We can declare the vector space to be real and say that the abstract basis vectors (J_+, J_-, J_3) have the commutation relation given above. This defines a real Lie algebra. More concretely we can define the real algebra $\mathfrak{sl}(2,\mathbb{R})$ of traceless 2×2 real matrices (naturally a real vector space).

▶ There is another way to construct another real algebra from $\mathfrak{sl}(2,\mathbb{C})$. As we have seen before, we can write the raising/lowering operators as a function of anti-hermitian matrices s_i, i.e.

$$\mathfrak{sl}(2,\mathbb{C}) := \{X \in \mathbb{C}^{2\times 2} \big| \ X = \alpha^i s_i\,, \quad \alpha^i \in \mathbb{C}\} \equiv \mathrm{span}_{\mathbb{C}}\{s_i\}_{i=1}^3 \tag{3.51}$$

with

$$s_1 = \frac{-i}{2}(J_+ + J_-)\,, \quad s_2 = \frac{1}{2}(J_- - J_+)\,, \quad s_3 = \frac{-i}{2}\tilde{J}_3\,.$$

and

$$[s_i, s_j] = \epsilon_{ijk} s_k$$

► Declaring the s_k to be basis vector of a real vector space we get the familiar $\mathfrak{su}(2)$ algebra, described as the algebra of 2×2 anit-hermitian matrices

$$\mathfrak{su}(2) = \text{span}_{\mathbb{R}}\{s_i\}_{i=1}^{3}$$

Any representation T of $\mathfrak{sl}(2, \mathbb{C})$ is also a representation of $\mathfrak{su}(2)$; all we need to do is restrict the domain of T.

★ This is why we used the ladder operators to construct irreps in $\mathfrak{so}(3)$ (which is isomorphic to $\mathfrak{su}(2)$), even though they did not belong to the $\mathfrak{so}(3)$ Lie algebra.

★ The ladder operators belong to the complexified algebra $\mathfrak{sl}(2, \mathbb{C})$. Therefore, we where secretly working with this algebra the whole time.

★ Although the matrices have complex entries, the vector space is naturally real, complex multiplication would ruin anti-hermiticity.

★ The real algebras $\mathfrak{sl}(2, \mathbb{R})$ and $\mathfrak{su}(2)$ are not isomorphic over the real, they are the two real forms associated with $\mathfrak{sl}(2, \mathbb{C})$.

3.7 The direct product space

Direct product

Given a $m \times n$ matrix A and a $p \times q$ matrix B, their **direct product** (**tensor product**, **Kronecker product**) $A \otimes B$ is a matrix with dimensions $mp \times nq$ and amounts to:

Replacing A_{ij} by the matrix $A_{ij}B$.

▶ We are interested in looking at the product of irreducible representations. Those are obtained using the tensor product of the representations, i.e. $D = D_1 \otimes D_2$.

▶ As an example, we select here the product space $D^{(1/2)} \otimes D^{(1)}$. The basis vectors of this space are defined as

$$\left|\frac{1}{2}, \frac{1}{2}\right\rangle \otimes |1, m\rangle = \begin{pmatrix} |1, m\rangle \\ \vec{0} \end{pmatrix} \quad \text{and} \quad \left|\frac{1}{2}, -\frac{1}{2}\right\rangle \otimes |1, m\rangle = \begin{pmatrix} \vec{0} \\ |1, m\rangle \end{pmatrix} . \tag{3.52}$$

in a more explicit way

$$\left|\frac{1}{2}, \frac{1}{2}\right\rangle \otimes |1, 1\rangle = \hat{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} , \quad \left|\frac{1}{2}, \frac{1}{2}\right\rangle \otimes |1, 0\rangle = \hat{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} , \tag{3.53}$$

$$\left|\frac{1}{2}, \frac{1}{2}\right\rangle \otimes |1, -1\rangle = \hat{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} , \quad \left|\frac{1}{2}, -\frac{1}{2}\right\rangle \otimes |1, 1\rangle = \hat{e}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} , \tag{3.54}$$

$$\left|\frac{1}{2},-\frac{1}{2}\right\rangle \otimes |1,0\rangle = \hat{e}_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad , \quad \left|\frac{1}{2},-\frac{1}{2}\right\rangle \otimes |1,-1\rangle = \hat{e}_6 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{3.55}$$

Using the matrix representations $D^{(1/2)}(U(\vec{n}))$ and $D^{(1)}(U(\vec{n}))$ of a group element $U(\vec{n}) \in SU(2)$ (in the basis previously defined), we let $U(\vec{n})$ in the product space $D^{(1/2)} \otimes D^{(1)}$ be represented by

$$D(U(\vec{n})) = \begin{pmatrix} [D^{(1/2)}(U(\vec{n}))]_{11} D^{(1)}(U(\vec{n})) & [D^{(1/2)}(U(\vec{n}))]_{12} D^{(1)}(U(\vec{n})) \\ [D^{(1/2)}(U(\vec{n}))]_{21} D^{(1)}(U(\vec{n})) & [D^{(1/2)}(U(\vec{n}))]_{22} D^{(1)}(U(\vec{n})) \end{pmatrix} \tag{3.56}$$

The above matrix represents a 6×6 matrix, because $D^{(1)}(U(\vec{n}))$ stands for a 3×3 matrix.

▶ The way this matrix acts in the basis vectors is given by

★

$$\begin{aligned} D(U(\vec{n}))\left|\frac{1}{2},\frac{1}{2}\right\rangle \otimes |1,m\rangle =& D(U(\vec{n}))\begin{pmatrix} |1,m\rangle \\ \vec{0} \end{pmatrix} \\ =& \begin{pmatrix} [D^{(1/2)}(U(\vec{n}))]_{11} D^{(1)}(U(\vec{n}))\,|1,m\rangle \\ [D^{(1/2)}(U(\vec{n}))]_{21} D^{(1)}(U(\vec{n}))\,|1,m\rangle \end{pmatrix} \\ =& D^{(1/2)}(U(\vec{n}))\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes D^{(1)}(U(\vec{n}))\,|1,m\rangle \\ =& \left\{ D^{(1/2)}(U(\vec{n}))\left|\frac{1}{2},\frac{1}{2}\right\rangle \right\} \otimes \left\{ D^{(1)}(U(\vec{n}))\,|1,m\rangle \right\} \end{aligned}$$

★

$$
\begin{aligned}
D(U(\vec{n}))\left|\frac{1}{2},-\frac{1}{2}\right\rangle \otimes |1,m\rangle =& D(U(\vec{n}))\begin{pmatrix}\vec{0}\\ |1,m\rangle\end{pmatrix}\\
=&\begin{pmatrix}[D^{(1/2)}(U(\vec{n}))]_{12}D^{(1)}(U(\vec{n}))\,|1,m\rangle\\ [D^{(1/2)}(U(\vec{n}))]_{22}D^{(1)}(U(\vec{n}))\,|1,m\rangle\end{pmatrix}\\
=&D^{(1/2)}(U(\vec{n}))\begin{pmatrix}0\\1\end{pmatrix}\otimes D^{(1)}(U(\vec{n}))\,|1,m\rangle\\
=&\left\{D^{(1/2)}(U(\vec{n}))\left|\frac{1}{2},-\frac{1}{2}\right\rangle\right\}\otimes\left\{D^{(1)}(U(\vec{n}))\,|1,m\rangle\right\}
\end{aligned}
$$

► As is known from theorem 9 in the discrete part of this course (or the gray text above) the product state transforms according to the direct product representation.

$$
\begin{gathered}
D(U(\vec{n})) = D^{(1/2)}(U(\vec{n})) \otimes D^{(1)}(U(\vec{n}))\\
\Longrightarrow\\
e^{-i\vec{n}.d(\vec{J})} = e^{-i\vec{n}\cdot d^{(1/2)}(\vec{J})} \otimes e^{-i\vec{n}\cdot d^{(1)}(\vec{J})}
\end{gathered}
\tag{3.57}
$$

► Differentiating we find for the generators of the direct product space

$$
\begin{aligned}
d(J_i) =& i\left.\frac{\partial D(U(\vec{n}))}{\partial n_i}\right|_{\vec{n}=0} = i\left[\left.\frac{\partial D^{(1/2)}(U(\vec{n}))}{\partial n_i}\right|_{\vec{n}=0}\right]\otimes D^{(1)}(U(\vec{n}=0))\\
&+iD^{(1/2)}(U(\vec{n}=0))\otimes\left[\left.\frac{\partial D^{(1)}(U(\vec{n}))}{\partial n_i}\right|_{\vec{n}=0}\right]\\
=&d^{(1/2)}(J_i)\otimes\mathbb{I}_3+\mathbb{I}_2\otimes d^{(1)}(J_i)
\end{aligned}
\tag{3.58}
$$

► In general the representation $d(A)$ in the product space $D^{(1/2)}\otimes D^{(1)}$ of an arbitrary element X of the Lie-algebra $\mathfrak{su}(2)$ is given by the following transformation rule

$$
d(X)\left|\frac{1}{2},m_1\right\rangle\otimes|1,m_2\rangle = \left[d^{(1/2)}(X)\left|\frac{1}{2},m_1\right\rangle\right]\otimes|1,m_2\rangle+\left|\frac{1}{2},m_1\right\rangle\otimes\left[d^{(1)}(X)\,|1,m_2\rangle\right]
$$

Generalizations

In general, a tensor product between Lie group representations is defined by

$$
\left(D^1\otimes D^2\right)(t)\,|\psi_1\rangle\otimes|\psi_2\rangle := D^1(t)\,|\psi_1\rangle\otimes D^2(t)\,|\psi_2\rangle \tag{3.59}
$$

For the Lie algebra representations we have

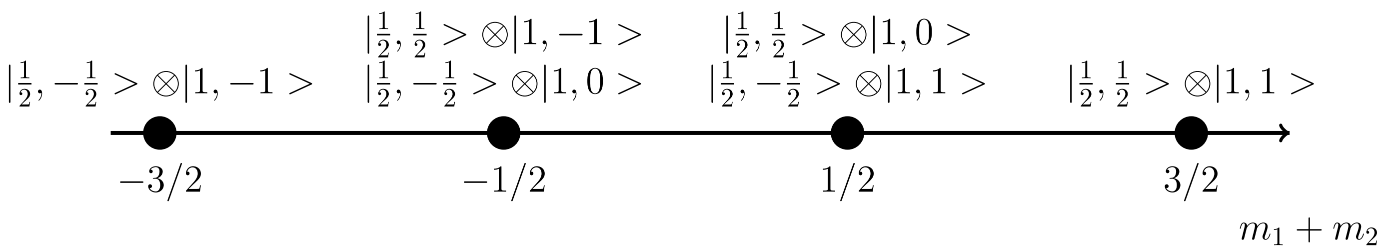
$$
\left(d^1\otimes d^2\right)(X)\,|\psi_1\rangle\otimes|\psi_2\rangle := d^1(X)\,|\psi_1\rangle\otimes|\psi_2\rangle+|\psi_1\rangle\otimes d^2(X)\,|\psi_2\rangle \tag{3.60}
$$

▶ The proof is by differentiating, as for the special case above.

▶ Since both $d^{(1/2)}(J_3)$ and $d^{(1)}(J_3)$ change the state on which they are acting only by multiplying with m, the operator $d(J_3)$ is also diagonal with eigenvalues $m_1 + m_2$

$$d(J_3)\left[\left|\frac{1}{2}, m_1\right\rangle \otimes |1, m_2\rangle\right] = (m_1 + m_2)\left|\frac{1}{2}, m_1\right\rangle \otimes |1, m_2\rangle \tag{3.61}$$

▶ We call the eigenvalues of J_3 **weights** and we draw what is known as a **weight diagram**



▶ There exists different eigenvectors of $d(J_3)$ with the same eigenvalue. This is an indication that the representation of $SU(2)$ in the product space $D^{(1/2)} \otimes D^{(1)}$ must be reducible, since the states in an irreducible representation are reached only once when acting with $J_\pm$.

▶ Let us look at the decomposition of the product representation:

★ We first determine the matrix representation of J^2 in the product space, which for arbitrary representations neither is proportional to the identity, nor diagonal. Diagonalization of J^2 leads to another basis of the product space, in which the matrix representations of the elements of the Lie-algebra have a block diagonal form.

★ Using $J^2 = [d(J_3)]^2 + d(J_3) + d(J_-)d(J_+)$ and

$$d(J_3) = \text{diagonal}[3/2, 1/2, -1/2, 1/2, -1/2, -3/2] \tag{3.62}$$

(from (3.61) and the basis vectors), we just need to find the representations of $d(J_\pm)$.

★ For $d(J_-)$ we find

$$\begin{aligned} d(J_-) =& d^{(1/2)}(J_-) \otimes \mathbb{I}_3 + \mathbb{I}_2 \otimes d^{(1)}(J_-) \\ =& \begin{pmatrix} \mathbb{O}_3 & \mathbb{O}_3 \\ \mathbb{I}_3 & \mathbb{O}_3 \end{pmatrix} + \begin{pmatrix} d^{(1)}(J_-) & \mathbb{O}_3 \\ \mathbb{O}_3 & d^{(1)}(J_-) \end{pmatrix} \\ =& \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & \sqrt{2} & 0 \end{pmatrix} \end{aligned}$$

For $d(J_+)$ we get the transposed matrix.

★ We then find for J^2

$$J^2 = \begin{pmatrix} \frac{15}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{11}{4} & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & \frac{7}{4} & 0 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & \frac{7}{4} & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & \frac{11}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{15}{4} \end{pmatrix} \tag{3.63}$$

As expected the matrix is not even diagonal!

★ Looking at such a matrix we find that 2 basis vectors out of the 6 $\hat{e}_i$ are eigenvectors of J^2, i.e.

$$\begin{aligned} J^2 \hat{e}_{1,6} =& \frac{15}{4} \hat{e}_{1,6} = \frac{15}{4} \left| \frac{1}{2}, \pm\frac{1}{2} \right\rangle \otimes |1, \pm 1\rangle \\ =& \frac{3}{2}\left(\frac{3}{2}+1\right) \left| \frac{3}{2}, \pm\frac{3}{2} \right\rangle \quad \text{fitting with } d(J_3) = \pm\frac{3}{2} \end{aligned} \tag{3.64}$$

★ Note that the "highest" state will only appear once

► We now might search for other eigenvectors of J^2. A more elegant way is to start from one basis vector and construct the complete basis for an irrep by repeatedly applying the raising or lowering operator.

★ Let us start with $\left|\frac{3}{2},\frac{3}{2}\right\rangle_4$ (the $\rangle_4$ is there since we know there will be 4 states with $m = 3/2, 1/2, -1/2, -3/2$) and apply the lowering operator

$$d^{(3/2)}(J_-)\left|\frac{3}{2},\frac{3}{2}\right\rangle_4 = \sqrt{\frac{3}{2}\left(\frac{3}{2}+1\right)-\frac{3}{2}\left(\frac{3}{2}-1\right)}\left|\frac{3}{2},\frac{1}{2}\right\rangle_4 = \sqrt{3}\left|\frac{3}{2},\frac{1}{2}\right\rangle_4 \tag{3.65}$$

Applying $d(J_-)$ to the product vector we get

$$\begin{aligned} d(J_-)\left[\left|\frac{1}{2},\frac{1}{2}\right\rangle\otimes|1,1\rangle\right] &= \left(d^{(1/2)}(J_-)\left|\frac{1}{2},\frac{1}{2}\right\rangle\right)\otimes|1,1\rangle + \left|\frac{1}{2},\frac{1}{2}\right\rangle\otimes\left(d^{(1)}(J_-)|1,1\rangle\right) \\ &= \left|\frac{1}{2},-\frac{1}{2}\right\rangle\otimes|1,1\rangle + \sqrt{2}\left|\frac{1}{2},\frac{1}{2}\right\rangle\otimes|1,0\rangle \end{aligned}$$

Therefore, we identify

$$\left|\frac{3}{2},\frac{1}{2}\right\rangle_4 = \sqrt{\frac{1}{3}}\left|\frac{1}{2},-\frac{1}{2}\right\rangle\otimes|1,1\rangle + \sqrt{\frac{2}{3}}\left|\frac{1}{2},\frac{1}{2}\right\rangle\otimes|1,0\rangle \tag{3.66}$$

The eigenvalue of $d(J_3)$ for the vector on the right-hand side is $1/2$. Using the matrix representation of J^2 we find that this eigenvector has eigenvalue $15/4$ as expected.

★ We apply $d^{(3/2)}(J_-)$ (left) and $d(J_-)$ (right) once more and get

$$\begin{aligned} d^{(3/2)}(J_-)\left|\frac{3}{2},\frac{1}{2}\right\rangle_4 &= d(J_-)\left[\sqrt{\frac{1}{3}}\left|\frac{1}{2},-\frac{1}{2}\right\rangle\oplus|1,1\rangle + \sqrt{\frac{2}{3}}\left|\frac{1}{2},\frac{1}{2}\right\rangle\otimes|1,0\rangle\right] \\ \Leftrightarrow\left|\frac{3}{2},-\frac{1}{2}\right\rangle_4 &= \sqrt{\frac{2}{3}}\left|\frac{1}{2},-\frac{1}{2}\right\rangle\otimes|1,0\rangle + \frac{1}{\sqrt{3}}\left|\frac{1}{2},\frac{1}{2}\right\rangle\otimes|1,-1\rangle \end{aligned} \tag{3.67}$$

★ Therefore, using the lowering operator we have found the basis of a 4 dimensional irrep of $SU(2)$ ($j = 3/2$)

$$\begin{aligned} \hat{e}'_1 &= \left|\frac{3}{2},\frac{3}{2}\right\rangle_4 = \left|\frac{1}{2},\frac{1}{2}\right\rangle\otimes|1,1\rangle \\ \hat{e}'_2 &= \left|\frac{3}{2},\frac{1}{2}\right\rangle_4 = \sqrt{\frac{1}{3}}\left|\frac{1}{2},-\frac{1}{2}\right\rangle\otimes|1,1\rangle + \sqrt{\frac{2}{3}}\left|\frac{1}{2},\frac{1}{2}\right\rangle\otimes|1,0\rangle \\ \hat{e}'_3 &= \left|\frac{3}{2},-\frac{1}{2}\right\rangle_4 = \sqrt{\frac{2}{3}}\left|\frac{1}{2},-\frac{1}{2}\right\rangle\otimes|1,0\rangle + \frac{1}{\sqrt{3}}\left|\frac{1}{2},\frac{1}{2}\right\rangle\otimes|1,-1\rangle \\ \hat{e}'_4 &= \left|\frac{3}{2},-\frac{3}{2}\right\rangle_4 = \left|\frac{1}{2},-\frac{1}{2}\right\rangle\otimes|1,-1\rangle \end{aligned} \tag{3.68}$$

★ We are then left with a subspace of the product space with dimension 6-4=2. So we have (at most) $j = 1/2$. The basis vectors that have $d(J_3) = 1/2$ are

$$\left|\frac{1}{2},\frac{1}{2}\right\rangle \otimes |1,0\rangle \quad \text{and} \quad \left|\frac{1}{2},-\frac{1}{2}\right\rangle \otimes |1,1\rangle \,. \tag{3.69}$$

The 4-plet has one combination of such vectors, therefore we must find the orthogonal one, i.e.

$$\begin{aligned} &{}_4\!\left\langle\frac{3}{2},\frac{1}{2}\right| \left[a\left|\frac{1}{2},\frac{1}{2}\right\rangle \otimes |1,0\rangle + b\left|\frac{1}{2},-\frac{1}{2}\right\rangle \otimes |1,1\rangle\right] = 0 \\ &\Leftrightarrow a\sqrt{\frac{2}{3}} + b\sqrt{\frac{1}{3}} = 0 \qquad \underbrace{\Rightarrow}_{\text{(normalizing)}} \qquad \begin{cases} a = \sqrt{\frac{1}{3}} \\ b = -\sqrt{\frac{2}{3}} \end{cases} \end{aligned} \tag{3.70}$$

Therefore

$$\left|\frac{1}{2},\frac{1}{2}\right\rangle_2 = \sqrt{\frac{1}{3}}\left|\frac{1}{2},\frac{1}{2}\right\rangle \otimes |1,0\rangle - \sqrt{\frac{2}{3}}\left|\frac{1}{2},-\frac{1}{2}\right\rangle \otimes |1,1\rangle \tag{3.71}$$

This this is an eigenvector of J^2 with eigenvalue 3/4.

★ Applying the lowering operator to this basis vector we get

$$\begin{aligned} &d^{(1/2)}(J_-)\left|\frac{1}{2},\frac{1}{2}\right\rangle_2 = d(J_-)\left[\sqrt{\frac{1}{3}}\left|\frac{1}{2},\frac{1}{2}\right\rangle \otimes |1,0\rangle - \sqrt{\frac{2}{3}}\left|\frac{1}{2},-\frac{1}{2}\right\rangle \otimes |1,1\rangle\right] \\ \Leftrightarrow &\left|\frac{1}{2},-\frac{1}{2}\right\rangle_2 = \sqrt{\frac{2}{3}}\left|\frac{1}{2},\frac{1}{2}\right\rangle \otimes |1,-1\rangle - \sqrt{\frac{1}{3}}\left|\frac{1}{2},-\frac{1}{2}\right\rangle \otimes |1,0\rangle \end{aligned} \tag{3.72}$$

★ Therefore, we find the new basis for the doublet irrep

$$\begin{aligned} \hat{e}'_5 &= \left|\frac{1}{2},\frac{1}{2}\right\rangle_2 = \sqrt{\frac{1}{3}}\left|\frac{1}{2},\frac{1}{2}\right\rangle \otimes |1,0\rangle - \sqrt{\frac{2}{3}}\left|\frac{1}{2},-\frac{1}{2}\right\rangle \otimes |1,1\rangle \\ \hat{e}'_6 &= \left|\frac{1}{2},-\frac{1}{2}\right\rangle_2 = \sqrt{\frac{2}{3}}\left|\frac{1}{2},\frac{1}{2}\right\rangle \otimes |1,-1\rangle - \sqrt{\frac{1}{3}}\left|\frac{1}{2},-\frac{1}{2}\right\rangle \otimes |1,0\rangle \end{aligned} \tag{3.73}$$

★ In this new vector basis, i.e. $\hat{e}'_i$, the matrix representation of all elements of the algebra and hence of the group (by exponentiating) are matrices of the form of a direct sum of a 4×4 matrix ($d^{(3/2)}$) and a 2×2 ($d^{(1/2)}$), i.e.

$$d'(O) = \begin{pmatrix} d^{(3/2)}(O) & 0 \\ 0 & d^{(1/2)}(O) \end{pmatrix}, \quad J^2 = \begin{pmatrix} \frac{15}{4}\mathbb{I}_4 & 0 \\ 0 & \frac{3}{4}\mathbb{I}_2 \end{pmatrix} = \begin{pmatrix} J^2(j = \frac{3}{2}) & 0 \\ 0 & J^2(j = \frac{1}{2}) \end{pmatrix} \tag{3.74}$$

and

$$\begin{aligned} D'(U) &= \begin{pmatrix} D^{3/2}(U) & 0 \\ 0 & D^{(1/2)}(U) \end{pmatrix}, \\ D' = \ D^{(1/2)} \otimes D^{(1)} &= D^{(3/2)} \oplus D^{(1/2)} \\ \mathbf{2} \otimes \mathbf{3} &= \mathbf{4} \oplus \mathbf{2} \end{aligned} \tag{3.75}$$

Generalizations

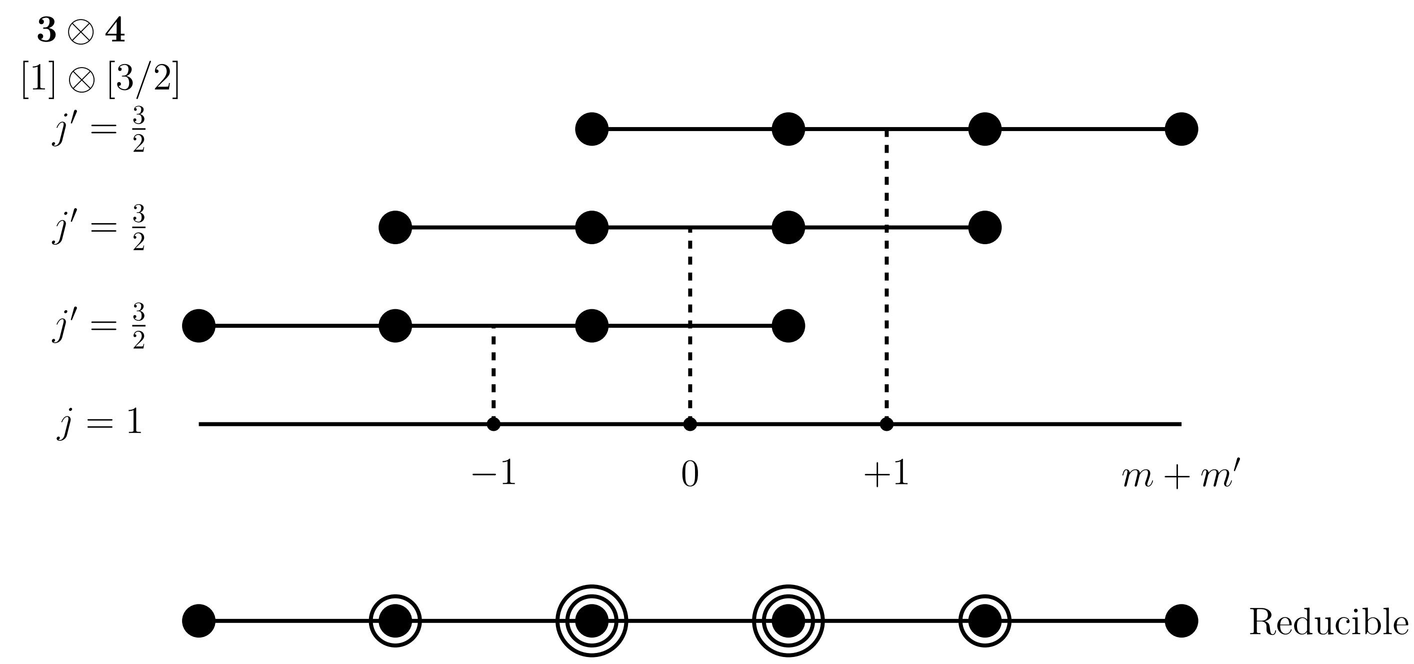
The tensor product of irreducible representations can be expanded as a direct sum of irreducible representations

$$D^{(j_1)} \otimes D^{(j_2)} = \bigoplus_j c_j D^{(j)} \tag{3.76}$$

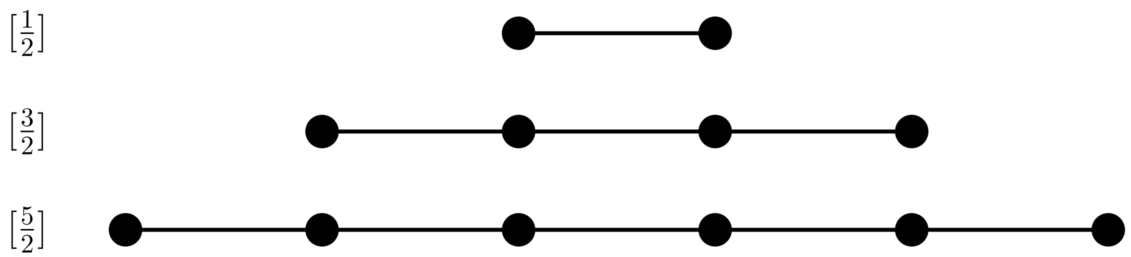
where c_j represent the multiplicity of that representation. This is called the **Clebsch-Gordan series**.

► We can use the graphical representation in order to find the direct products of representations. Take two $SU(2)$ representations, one with j and the other with j'. The product of such representations will give us a representation that is $(2j + 1)(2j' + 1)$ dimensional, but reducible. Since the total m will be the sum $m_1 + m_2$ we can find the possible states by the following steps:

- We draw first the j weight diagram (see figure below)
- We then draw repeatedly the j' diagram so it has its center appearing in each point of the j diagram



The $m + m'$ value $5/2$ appears once, $3/2$ twice, and so on. Preserving the number of points and the values, we arrange the points in a symmetric way as **multiplets** on different levels



Which again has one $5/2$, two $3/2$, and so on. We thus have found the Clebsch-Gordan series:

$$[1] \otimes [3/2] = [5/2] \oplus [3/2] \oplus [1/2] \tag{3.77}$$

This tells us what states to expect, but it does not give us the reduced basis. Also, note that the dimension counting works: $3 \times 4 = 6 + 4 + 2$.

▶ For the tensor product of two irreducible representations $D^{(j_1)}$ and $D^{(j_2)}$ of $\mathsf{SU}(2)$ (or $\mathsf{SO}(3)$ for $j \in \mathbb{N}$) we have the following decomposition

$$D^{(j_1)} \otimes D^{(j_2)} = \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} D^{(j)} \tag{3.78}$$

in unit steps.

▶ To show this we can use the characters of the representations:

★ We use the fact that the character of a direct product of representations is the product of the characters of the individual representations (theorem 8 in the discrete part of this course)

$$\chi^{(j_1)}(D^{(j_1)}(\hat{z},\alpha))\chi^{(j_2)}(D^{(j_2)}(\hat{z},\alpha)) = \sum_j c_j \chi^{(j)}(D^{(j)}(\hat{z},\alpha)) \tag{3.79}$$

where c_j is the number of times rep. j appear. We show below that c_j is 1.

★ For an individual representation we have

$$\begin{aligned}
\chi^j(D^{(j)(\hat{z},\alpha)}) = \sum_{m=-j}^{j} e^{-im\alpha} &= \sum_{m=0}^{j} e^{-im\alpha} + \sum_{m=0}^{j} e^{im\alpha} - 1 \\
&= \frac{1-e^{-i\alpha(j+1)}}{1-e^{-i\alpha}} + \frac{1-e^{i\alpha(j+1)}}{1-e^{i\alpha}} - 1 \\
&= \frac{e^{i\alpha/2}-e^{-i\alpha(j+1/2)}}{e^{i\alpha/2}-e^{-i\alpha/2}} + \frac{e^{-i\alpha/2}-e^{i\alpha(j+1/2)}}{e^{-i\alpha/2}-e^{i\alpha/2}} - 1 \\
&= \frac{e^{i\alpha(j+1/2)}-e^{-i\alpha(j+1/2)}}{e^{i\alpha/2}-e^{-i\alpha/2}} = \frac{\sin\alpha(\frac{1}{2}+j)}{\sin\frac{\alpha}{2}}
\end{aligned}$$

★ Taking $j_1 \geq j_2$ (no loss of generality)

$$\begin{aligned}
\chi^{(j_1)}(D^{(j_1)}(\hat{z},\alpha))\chi^{(j_2)}(D^{(j_2)}(\hat{z},\alpha)) &= \left[\sum_{m=-j_1}^{j_1} e^{-im\alpha}\right]\left[\sum_{m=-j_2}^{j_2} e^{-im\alpha}\right] \\
&= \frac{e^{i(j_1+1/2)\alpha}-e^{-i(j_1+1/2)\alpha}}{2i\sin\alpha/2}\left[\sum_{m=-j_2}^{j_2} e^{-im\alpha}\right] \\
&= \frac{1}{2i\sin\frac{\alpha}{2}}\sum_{m=-j_2}^{j_2}\left(e^{i(j_1+m+\frac{1}{2})\alpha} - e^{-i(j_1-m+\frac{1}{2})\alpha}\right)
\end{aligned}$$

★ Since the sum over m is symmetric we can change the sign of m in the second term, we get

$$\begin{aligned}
\chi^{(j_1)}(D^{(j_1)}(\hat{z},\alpha))\chi^{(j_2)}(D^{(j_2)}(\hat{z},\alpha)) &= \frac{1}{2i\sin\frac{\alpha}{2}}\sum_{m=-j_2}^{j_2}\left(e^{i(j_1+m+\frac{1}{2})\alpha} - e^{-i(j_1+m+\frac{1}{2})\alpha}\right) \\
&= \frac{1}{2i\sin\frac{\alpha}{2}}\sum_{j=j_1-j_2}^{j_1+j_2}\left(e^{i(j+\frac{1}{2})\alpha} - e^{-i(j+\frac{1}{2})\alpha}\right) \\
&= \sum_{j=j_1-j_2}^{j_1+j_2}\frac{\sin(j+\frac{1}{2})\alpha}{\sin\frac{\alpha}{2}} \equiv \sum_{j=j_1-j_2}^{j_1+j_2}\chi^{(j)}(D^{(j)}(\hat{z},\alpha))
\end{aligned}$$

★ Therefore, $c_j = 1$. The only difference between $\mathsf{SO}(3)$ and $\mathsf{SU}(2)$ in this proof is in j being only integer or also half-integer, respectively.

► The procedure of working out the irreducible representations of the product representations can be summarized as follows:

(1) Start with the combination of states with the largest m, i.e. the eigenstate of $d^{(j_1+j_2)}(J_3)$ with eigenvalue $j_1 + j_2$ ($|j_1 + j_2, j_1 + j_2\rangle$);

(2) Use the lowering operator $d^{(j_1+j_2)}(J_-)$ to get all the other states in the same irreducible representation.

(3) Find the orthogonal combination to $|j_1 + j_2, j_1 + j_2 - 1\rangle$. This will now be the state $|j_1 + j_2 - 1, j_1 + j_2 - 1\rangle$. Then use the lowering operator to reach the other $j_1 + j_2 - 1$ states.

(4) Repeat these steps until you reach the state $||j_1 - j_2|, \cdot\rangle$.

► Although we can reduce $D^{(j_1)} \otimes D^{(j_2)}$ into irreducible components $\bigoplus D^{(j)}$, we still need to find an appropriate basis in order to express these matrices in a block diagonal form. We did this for a particular example in $\mathsf{SU}(2)$.

► The general identification between the basis vectors of the product space and the basis vectors of the final product irreps is given by the following theorem

Theorem:

The two bases $|j, m\rangle$ and $|j_1, m_1\rangle \otimes |j_2, m_2\rangle$ are related by:

$$|j, m\rangle = \sum_{m_1 m_2} C^{j_1 j_2 j}_{m_1 m_2 m} |j_1, m_1\rangle \otimes |j_2, m_2\rangle = \sum_{m_1 m_2} C^{j_1 j_2 j}_{m_1 m_2 m} |j_1, m_1; j_2, m_2\rangle$$

The coefficients $C^{j_1 j_2 j}_{m_1 m_2 m}$ are called **Clebsch-Gordan coefficients**.

► The CG coefficients can be determined through the use of the orthogonality of $|j_i, m_i\rangle$, i.e.

$$C^{j_1 j_2 j}_{m_1 m_2 m} = \langle j_1, m_1; j_2, m_2 \,| j, m\rangle \tag{3.80}$$

Since both bases $\{|j, m\rangle\}$ and $\{|j_1, m_1\rangle \otimes |j_2, m_2\rangle\}$ are orthonormal by construction, the transformation in Eq. (3.80) must be unitary, and its inverse is

$$|j_1, m_1; j_2, m_2\rangle = \sum_{j=|j_1-j_2|}^{j_1+j_2} \sum_{m=-j}^{j} |j, m\rangle \langle j, m|\, j_1, m_1; j_2, m_2\rangle \tag{3.81}$$

For a variety of historical reasons, these coefficients can also be written as

$$
\begin{aligned}
C^{j_1 j_2 j}_{m_1 m_2 m} =&(j_1, j_2; m_1, m_2|j, m) \quad \text{(I will use this!)}\\
=&(j, j_1, j_2|m, m_1, m_2)\\
=&(-1)^{j_1-j_2+m}\sqrt{2j+1}\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{pmatrix}
\end{aligned}
$$

In our previous example of $\mathbf{2} \otimes \mathbf{3} = \mathbf{4} \oplus \mathbf{2}$ we had the following Clebsch-Gordan coefficients

★ 4-plet ($\left|\frac{3}{2}, m\right\rangle$)	★ 2-plet ($\left|\frac{1}{2}, m\right\rangle$)
$\left(\frac{1}{2}, 1; \frac{1}{2}, 1 \,\middle|\, \frac{3}{2}, \frac{3}{2}\right) = 1$	$\left(\frac{1}{2}, 1; \frac{1}{2}, 0 \,\middle|\, \frac{1}{2}, \frac{1}{2}\right) = \sqrt{\frac{1}{3}}$
$\left(\frac{1}{2}, 1; -\frac{1}{2}, 1 \,\middle|\, \frac{3}{2}, \frac{1}{2}\right) = \sqrt{\frac{1}{3}}$	$\left(\frac{1}{2}, 1; -\frac{1}{2}, 1 \,\middle|\, \frac{1}{2}, \frac{1}{2}\right) = -\sqrt{\frac{2}{3}}$
$\left(\frac{1}{2}, 1; \frac{1}{2}, 0 \,\middle|\, \frac{3}{2}, \frac{1}{2}\right) = \sqrt{\frac{2}{3}}$	$\left(\frac{1}{2}, 1; \frac{1}{2}, -1 \,\middle|\, \frac{1}{2}, -\frac{1}{2}\right) = \sqrt{\frac{2}{3}}$
$\left(\frac{1}{2}, 1; -\frac{1}{2}, 0 \,\middle|\, \frac{3}{2}, -\frac{1}{2}\right) = \sqrt{\frac{2}{3}}$	$\left(\frac{1}{2}, 1; -\frac{1}{2}, 0 \,\middle|\, \frac{1}{2}, -\frac{1}{2}\right) = -\sqrt{\frac{1}{3}}$
$\left(\frac{1}{2}, 1; \frac{1}{2}, -1 \,\middle|\, \frac{3}{2}, -\frac{1}{2}\right) = \sqrt{\frac{1}{3}}$	
$\left(\frac{1}{2}, 1; -\frac{1}{2}, -1 \,\middle|\, \frac{3}{2}, -\frac{3}{2}\right) = 1$	

There are many general expressions for the CG coefficients, non of them is easy. The one due to Van der Waerden is the most symmetric one. Its derivation is highly non-trivial, and I will not even try to derive the result

$$
\begin{aligned}
&\langle jm|\, j_1 m_1; j_2 m_2 \,\rangle = \delta_{m,m_1+m_2}\Delta(j_1, j_2, j)\\
&\times \sum_t (-1)^t \frac{[(2j+1)(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!(j+m)!(j-m)!]^{1/2}}{t!(j_1+j_2-j-t)!(j_1-m_1-t)!(j_2+m_2-t)!}\\
&\qquad \times \frac{1}{(j-j_2+m_1+t)!(j-j_1-m_2+t)!}
\end{aligned}
\tag{3.82}
$$

where

$$
\Delta(j_1, j_2, j) = \left[\frac{(j_1+j_2-j)!(j_1-j_2+j)!(-j_1+j_2+j)!}{(j_1+j_2+j+1)!}\right]^{1/2} \tag{3.83}
$$

and the sum runs over all values of t that do not lead to negative factorials. This expressions are valid for integer and half-integer indices (i.e. can be used for $SO(3)$ and $SU(2)$).

3.8 The reality property of SU(2) representations

▶ Given a Lie algebra

$$[T_a, T_b] = c_{abc}T_c,$$

the representation $\bar{T}_a = -T_a^T$ also satisfies the Lie algebra. To see this, we transpose the above equation

$$[T_a, T_b]^T = c_{abc}T_c^T \quad \Leftrightarrow \quad -[T_a^T, T_b^T] = c_{abc}T_c^T \quad \Leftrightarrow \quad [(-T_a^T), (-T_b^T)] = c_{abc}(-T_c^T)$$

For anti-hermitian generators, T_a, we get $\bar{T}_a = T_a^*$ since $T_a^\dagger = T_a^{*T} = -T_a \implies T_a^T = -T_a^*$.

▶ If we make the physicist choice, hermitian generators, like we did in $\mathsf{SO}(3)$ and $\mathsf{SU}(2)$, the algebra equation has a factor i

$$[T_a, T_b] = if_{abc}T_c$$

and conjugating this, we get

$$[T_a, T_b]^* = -if_{abc}T_c^* \quad \Leftrightarrow \quad [T_a^*, T_b^*] = if_{abc}(-T)_c^* \quad \Leftrightarrow \quad [(-T_a^*), (-T_b^*)] = if_{abc}(-T_c^*)$$

thus $\bar{T}_a = -T_a^*$ is also a representation. This corresponds to the complex conjugated representation since under complex conjugation

$$e^{-i\alpha_a T_a} \to e^{+i\alpha_a T_a^*} = e^{-i\alpha_a (-T_a)^*} = e^{-i\alpha_a \bar{T}_a}.$$

▶ With the hermitian generator choice, irreducible representations of a Lie algebra can be classified in three types:

★ **Real representation** $\quad \bar{T}_a = -(T_a)^* = ST_aS^{-1}\,, \quad$ with $\quad S^T = S$
In general the representation and the conjugated representation are related by a symmetric similarity transformation. A special case is of course when all T_a are real.

★ **Pseudoreal representation** $\bar{T}_a = -(T_a)^* = ST_aS^{-1}\,, \quad$ with $\quad S^T = -S$

★ **Complex representation** $\bar{T}_a = -(T_a)^* \neq ST_aS^{-1}$

▶ In the case of the $\mathsf{SU}(2)$ group we only have real and pseudoreal representations. For example, for the $\mathfrak{su}(2)$ algebra we have:

- For $j = 1/2$ (doublet) the generators are given by

$$d^{(1/2)}(J_a) = \frac{\sigma_a}{2} \quad \Rightarrow -\left[d^{(1/2)}(J_a)\right]^* = \sigma_2 d^{(1/2)}(J_a)\sigma_2^{-1} \tag{3.84}$$

Since $\sigma_2^T = -\sigma_2$ the representation is pseudoreal.

- For $j = 1$ (triplet) the generators are given by

$$\begin{aligned} d^{(1)}(J_a) &= \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \end{pmatrix}, \frac{i}{\sqrt{2}} \begin{pmatrix} & -1 & \\ 1 & & -1 \\ & 1 & \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix} \right\} \\ \Rightarrow -\left[d^{(1)}(J_a)\right]^* &= \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix} d^{(1)}(J_a) \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix} \end{aligned} \tag{3.85}$$

The matrix is symmetric therefore the representation is real.

Chapter 4

The power of the SUN

More details about $SU(N)$ groups and $\mathfrak{su}(N)$ algebras

4.1 More general concepts

▶ **Simple Lie group:** A Lie group is called **simple** if is not abelian and does not posses a continuous (hence Lie type) invariant subgroup.
Ex: $SU(2), SO(3), SU(N)$ are simple, whereas $U(N)$ is not.

▶ **Semi-simple Lie group:** A Lie group is called **semi-simple** if is not abelian and it does not possess a continuous abelian invariant subgroup. It can contain a continuous invariant subgroup, but not continuous abelian invariant.

▶ If one can linearly combine M infinitesimal generators $\widehat{A}_l$ $(M < N)$ out of the N infinitesimal generators $\widehat{G_i}$ of a Lie group, such that

$$[\widehat{A}_l, \widehat{A_k}] \in \{\widehat{A_i}\} \tag{4.1}$$

the M infinitesimal generators $\widehat{A}_l$ of the invariant subgroup form a **subalgebra** of the original Lie algebra. If

$$[\widehat{G_i}, \widehat{A_k}] \in \{\widehat{A}_l\} \tag{4.2}$$

holds, such a subalgebra is called an **invariant subalgebra** or an **ideal**, and the Lie group processes an invariant subgroup. This can be seen by exponentiating, using the Baker-Campbell-Hausdorff formula.

▶ **Simple Lie algebra:** A Lie algebra $\mathfrak{g}$ is called **simple** if it is not abelian and it does not possess an ideal apart from the null ideal $\{0\}$ or $\mathfrak{g}$.

▶ **Semi-simple Lie algebra:** A Lie algebra is called semi-simple if is not abelian and it does not possess and abelian ideal. Thus (4.2) may hold but not all commutators $[\widehat{A_i}, \widehat{A_j}]$ in (4.1) are allowed to vanish.

▶ Every semi-simple Lie algebra $\mathfrak{g}$ is a direct sum of a set of simple Lie algebras, i.e. there exists a set of invariant simple subalgebras $\mathfrak{g}_1$, $\mathfrak{g}_1, \ldots \mathfrak{g}_k$ such that

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k \,.$$

This means that every semi-simple Lie group G is the direct product of simple Lie groups G_i

$$\mathsf{G} = \mathsf{G}_1 \otimes \mathsf{G}_2 \otimes \cdots \otimes \mathsf{G}_k \,.$$

A matrix Lie group G is simple (semi-simple) if and only if its Lie algebra is simple (semi-simple)

▶ For example: the $\mathfrak{so}(3)$ algebra is given by

$$[L_i, L_j] = i\epsilon_{ijk} L_k \,.$$

There is no smaller set of generators L_i that is closed. Therefore, the algebra does not possess an ideal, it is simple. Consequently, $\mathsf{SO}(3)$ is a simple group. The same applies to $\mathsf{SU}(2)$.

4.2 The Lie algebras $\mathfrak{su}(N)$

4.2.1 Hermitian matrices

▶ As is well known Hermiticity is defined as

$$\lambda^\dagger = \lambda \quad \longleftrightarrow \quad \lambda^*_{ik} = \lambda_{ki}$$

If we have, for example, $\lambda_{32} = r + is$ and all other elements zero the matrix reads

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & r - is & \\ 0 & r + is & 0 & \vdots \\ 0 & & \cdots & 0 \end{pmatrix} = r \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \\ 0 & 1 & 0 & \vdots \\ 0 & & \cdots & 0 \end{pmatrix} + s \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & \\ 0 & i & 0 & \vdots \\ 0 & & \cdots & 0 \end{pmatrix}$$

▶ We then observe that any Hermitian matrix can be constructed by real linear combinations of the following basis elements

$$
\begin{pmatrix} 1 & 0 & & 0 \\ 0 & 0 & & \\ & & & \vdots \\ 0 & & \cdots & 0 \end{pmatrix}, \quad
\begin{pmatrix} 0 & 0 & & 0 \\ 0 & 1 & & \\ & & & \vdots \\ 0 & & \cdots & 0 \end{pmatrix}, \cdots, \quad
\begin{pmatrix} 0 & 0 & & 0 \\ 0 & 0 & & \\ & & & \vdots \\ 0 & & \cdots & 1 \end{pmatrix}
$$

$$
\begin{pmatrix} 0 & 1 & & 0 \\ 1 & 0 & & \\ & & & \vdots \\ 0 & & \cdots & 0 \end{pmatrix}, \quad
\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & & \\ 1 & & & \vdots \\ 0 & & \cdots & 0 \end{pmatrix}, \cdots, \quad
\begin{pmatrix} 0 & 0 & & 0 \\ 0 & 0 & & \vdots \\ & & 0 & 1 \\ 0 & \cdots & 1 & 0 \end{pmatrix} \tag{4.3}
$$

$$
\begin{pmatrix} 0 & -i & & 0 \\ i & 0 & & \\ & & & \vdots \\ 0 & & \cdots & 0 \end{pmatrix}, \quad
\begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & & \\ i & & & \vdots \\ 0 & & \cdots & 0 \end{pmatrix}, \cdots, \quad
\begin{pmatrix} 0 & 0 & & 0 \\ 0 & 0 & & \vdots \\ & & 0 & -i \\ 0 & \cdots & i & 0 \end{pmatrix}
$$

The last two lines of elements have only traceless matrices. The total number of matrices is N^2. We can replace the the first line by the matrices

$$
\begin{pmatrix} 1 & 0 & & 0 \\ 0 & -1 & & \\ & & & \vdots \\ 0 & & \cdots & 0 \end{pmatrix}, \quad
\begin{pmatrix} 0 & 0 & & 0 \\ 0 & 1 & & \\ & & -1 & \vdots \\ 0 & & \cdots & 0 \end{pmatrix}, \cdots, \quad
\begin{pmatrix} 0 & 0 & & 0 \\ 0 & 0 & & \vdots \\ & & 1 & 0 \\ 0 & \cdots & 0 & -1 \end{pmatrix} \tag{4.4}
$$

which are now traceless and reduces the number of matrices to $N^2 - 1$.

▶ Returning to the anti-hermitian generators, we note that the matrices of dimension N form a Lie algebra

$$
a^\dagger = -a : \quad \begin{cases} (\alpha a + \beta b)^\dagger = -(\alpha a + \beta b) & \text{linear combinations are anti-hermitian} \\ [a, b]^\dagger = -[a, b] & \text{commutator is anti-hermitian} \end{cases}
$$

It is named $\mathfrak{u}(N)$ because it generates the Lie group $SU(N)$.

▶ On one hand, it constitutes a vector space over the field of real numbers. On the other hand, the matrices of the real matrix algebra $\mathfrak{u}(N)$ contain complex elements.

▶ We can write the anti-hermitian matrices in terms of hermitian matrices

$$a = -\frac{i}{2}h \quad \text{with} \quad h^\dagger = h$$

▶ Now we restrict ourselves to anti-hermitian $N \times N$ matrices a_a, b_s, ... with vanishing traces. Of course, linear combinations of such matrices also have vanishing traces. Generally, the trace of the commutator of matrices vanishes because of the cyclic property. This, combined with the anti-hermiticity of the commutator, implies that the commutator stays in the algebra. Every anti-hermitian matrix with vanishing trace can be formed as a linear combination with real coefficients of the matrices

$$e_j = -\frac{i}{2}\lambda_j$$

with λ_j the traceless hermitian matrices presented before.

▶ Therefore, the basis elements e_j constitute a real Lie algebra, the $\mathfrak{su}(N)$ algebra. Sometimes, the hermitian matrices λ_i are called the generators of $\mathfrak{su}(N)$. Note, however, that these do, in principle, not form a Lie algebra, only e_i do.

▶ The Lie algebra $\mathfrak{su}(N)$ can also be constituted by operators. Their basis elements $\widehat{e_j}$ correspond one-to-one to the basis matrices e_j. We define the operators $\widehat{e_j}$ by the equation

$$\widehat{e_j}\,|\psi_k\rangle = \sum_{l=1}^{N} (e_j)_{lk}\,|\psi_l\rangle \ , \quad (e_j)_{ik} = \langle\psi_i|\,\widehat{e_j}\,|\psi_k\rangle$$

Since the matrices e_j are anti-hermitian, also the operator $\widehat{e_j}$ is anti-hermitian. We can, in a similar way as we did for the matrices, write the anti-hermitian operator in terms of an hermitian one

$$\widehat{e_j} = -\frac{i}{2}\widehat{\lambda_j}\,.$$

For a general operator $\widehat{x}$ we have

$$\widehat{x}\,|\psi_k\rangle = \sum_l \Gamma(\widehat{x})_{lk}\,|\psi_l\rangle$$

with $\Gamma(\widehat{x})$ the matrix representation of the operator $\widehat{x}$.

4.2.2 Structure constants of $\mathfrak{su}(N)$

▶ We have that

$$[e_i, e_j] = \sum_{l=1}^{n} C_{ikl} e_l \quad \text{or} \quad [\lambda_i, \lambda_j] = \sum_{l=1}^{n} C_{ikl} 2i\lambda_l\,. \tag{4.5}$$

▶ The scalar C_{ikl} is a structure constant of the real Lie algebra $\mathfrak{su}(N)$ with basis elements e_i (not λ_i!), and n is the dimension of the algebra.

► Because the algebra $\mathfrak{su}(N)$ is real, the structure constants are real. Otherwise the right hand side of the commutator of e_j's would not be purely anti-hermitian.

► For the hermitian matrices λ_i and λ_k in Eq. (4.3), the following relation holds

$$\mathrm{Tr}\,(\lambda_i \lambda_k) = 2\delta_{ik} \tag{4.6}$$

This relation is true for the diagonal matrices in Eq. (4.4) only when the algebra is $\mathfrak{su}(2)$ but for higher $\mathfrak{su}(N)$ it is not. However, the set of matrices in Eq. (4.4) can be replaced by a set with the same number of linearly independent matrices which are real linear combinations of the previous ones and which satisfy the above trace relation. From now on we use these set of matrices.

► We can find the structure constants using (4.5) by computing

$$\mathrm{Tr}\,([\lambda_i, \lambda_k]\lambda_l) = 2i \sum_m C_{ikm} \mathrm{Tr}(\lambda_m \lambda_l) = 2i \sum_m C_{ikm} 2\delta_{ml} = 4iC_{ikl}$$

Therefore

$$C_{ikl} = \frac{1}{4i} \mathrm{Tr}\,([\lambda_i, \lambda_k]\lambda_l) = \frac{1}{4i}(\mathrm{Tr}(\lambda_i \lambda_k \lambda_l) - \mathrm{Tr}(\lambda_k \lambda_i \lambda_l))$$

► Using the cyclic property of the trace we get

$$C_{ikl} = -C_{ilk}$$

i.e. any odd permutation of the indices of C_{ikl} changes the sign. Therefore, C_{ikl} is totally antisymmetric in all indices. Consequently, no index appears more than once in a non-vanishing structure constant.

4.2.3 The adjoint matrices and Killing form

► We saw in Sec. 1 that the Jacobi identity, $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$, was one of the conditions in order for the algebra to be a Lie algebra.

► The Jacobi identity is trivially satisfied by the properties of the matrix multiplication. However, when using it with the Lie algebra commutator result, i.e. the linear combination of generators, we got a condition for the structure constants (1.10)

$$0 = \sum_l (C_{ilm} C_{jkl} + C_{jlm} C_{kil} + C_{klm} C_{ijl})$$

► This is nothing more than a representation of the algebra in disguise. Let us see

$$\begin{aligned}
\sum_l (C_{ilm} C_{jkl} + C_{jlm} C_{kil} + C_{klm} C_{ijl}) &= 0 \\
\sum_l (C_{ilm} C_{jkl} - C_{jlm} C_{ikl} - C_{lkm} C_{ijl}) &= 0
\end{aligned}$$

Defining

$$[\mathsf{adj}(e_i)]_{jk} = C_{ikj}$$

we get

$$\begin{aligned}\sum_l \left(\mathsf{adj}(e_i)_{ml}\mathsf{adj}(e_j)_{lk} - \mathsf{adj}(e_j)_{ml}\mathsf{adj}(e_i)_{lk} - \mathsf{adj}(e_l)_{mk}C_{ijl}\right) =& 0\\ \left(\mathsf{adj}(e_i)\mathsf{adj}(e_j) - \mathsf{adj}(e_j)\mathsf{adj}(e_i) - \sum_l \mathsf{adj}(e_l)C_{ijl}\right)_{mk} =& 0\end{aligned}$$

Therefore,

$$[\mathsf{adj}(e_i), \mathsf{adj}(e_j)] = \sum_l C_{ijl}\mathsf{adj}(e_l)$$

Therefore, the structure constants C_{ijk} themselves form a representation of the Lie algebra, the **regular** or **adjoint representation**. The coefficients $(\mathsf{adj}(e_i))_{lk}$ are elements of the $n \times n$ matrix $\mathsf{adj}(e_i)$.

▶ Therefore, we could have written the general Lie algebra commutation relation as

$$[e_i, e_k] = \sum_l^n (\mathsf{adj}(e_i))_{lk} e_l$$

▶ Consider now the adjoint representation of a general element a in the Lie algebra

$$[a, e_k] = \sum_l^n (\mathsf{adj}(a))_{lk} e_l \quad \text{for } i = 1, \ldots, n.$$

From the commutator properties, the following relations hold

$$\begin{aligned}\text{Linear: } & \mathsf{adj}(\alpha a + \beta b) = \alpha\mathsf{adj}(a) + \beta\mathsf{adj}(b)\\ \text{Commutator: } & \mathsf{adj}([a, b]) = [\mathsf{adj}(a), \mathsf{adj}(b)]\end{aligned}$$

This tells us that the matrices $\{\mathsf{adj}(a), \mathsf{adj}(b), \ldots\}$ indeed constitute a representation of the algebra $\{a, b, \ldots\}$. When $a = e_i$ we get the previous result.

▶ The **Killing form** $g(a, b)$ corresponding to any two elements a and b of a Lie algebra is defined by

$$g(a, b) = \mathrm{Tr}\left(\mathsf{adj}(a)\mathsf{adj}(b)\right)$$

The Killing form is symmetric and bilinear in the elements a and b.

▶ The Killing form of the basis elements e_i and e_j of a Lie algebra is

$$\begin{aligned}g_{ij} = g(e_i, e_j) =& \mathrm{Tr}\left(\mathsf{adj}(e_i)\mathsf{adj}(e_j)\right) = \sum_l \left(\sum_k (\mathsf{adj}(e_i))_{lk}(\mathsf{adj}(e_j))_{kl}\right)\\ =& \sum_{lk} C_{ikl}C_{jlk}\end{aligned}$$

This matrix, g, with components g_{ij}, is known as the **Cartan metric**.

▶ For $\mathfrak{su}(N)$ algebra, using the orthonormal generators, and the antisymmetry of C_{ikl}, the Cartan metric is given by

$$g_{ij} = g(e_i, e_j) = -\sum_{l,k=1}^{n} C_{ikl}^2 \delta_{ij}$$

▶ Many of the structural properties of the Lie algebra can be read off from the Killing form. For example, it provides a criterion for determining if a Lie algebra $\mathfrak{g}$ is semi-simple:

★ **Theorem (Cartan):** A Lie algebra $\mathfrak{g}$ is semi-simple if and only if its Killing form (in the adjoint representation) is non-degenerate, i.e.

$$\det[g(e_i, e_j)] \neq 0\,.$$

▶ Furthermore, a semi-simple Lie algebra is compact if the Killing form in the adjoint representation is negative definite.

▶ It is sometimes awkward to work out the Killing form in the adjoint representation (since this can be quite large). However, for simple Lie algebras, the Killing form, generalized to other representations, is the same in any representation, up to an overall proportionality factor.

▶ For example:

★ The $\mathfrak{su}(2)$ Lie algebra can be generated by the three 2×2 anti-hermitian s_i matrices. In the adjoint representation $[\mathsf{adj}(e_i)]_{jk} = C_{ikj} = \epsilon_{ikj}$ these matrices take the form

$$d^{(1)}(s_1) = \mathsf{adj}(s_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad d^{(1)}(s_2) = \mathsf{adj}(s_2) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

$$d^{(1)}(s_3) = \mathsf{adj}(s_3) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

They can be related to the anti-hermitian versions of the generators in (2.71) and (2.74) via a similarity transformation. Defining $d^{(1')}(s_i) = -id^{(1)}(L_i)$, s.t.,

$$d^{(1')}(L_1) = \frac{-i}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad d^{(1')}(L_2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$
$$d^{(1')}(L_3) = -i \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

the two representations can be related by a similarity transformation, $d^{(1')}(L_i) = Sd^{(1)}(s_1)S^{-1}$ where

$$S = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{pmatrix}.$$

The adjoint representation of SU(2) and the spin one representation of SO(3) or SU(2) are thus equivalent.

The Killing form in the adjoint representation gives

$$g = -2\mathbb{I}_3$$

The Killing form is indeed negative definite, thus $\mathfrak{su}(2) \cong \mathfrak{so}(3)$ is a compact Lie algebra.

We can also evaluate the Killing form for the 2×2 representation. We get

$$g_{2d} = -\frac{1}{2}\mathbb{I}_3$$

Therefore, both Killing forms are proportional (with a real factor) as they should be for simple Lie algebras.

4.3 Introducing $SU(3)$

4.3.1 Generators and $\mathfrak{su}(3)$ algebra

▶ The $SU(3)$ group is defined by the set of matrices

$$\mathsf{SU}(3) := \left\{ g \in \mathsf{SL}(3;\mathbb{C}) \,|\, gg^\dagger = \mathbb{I}_3 \right\} \tag{4.7}$$

▶ As seen in the first exercise session, $\mathsf{SU}(3)$ is an 8 parameter group ⇒ 8 independent generators of the $\mathfrak{su}(3)$ algebra. (For raising/lowering operators we are secretly working with $\mathfrak{sl}(3;\mathbb{C})$.),
▶ The Gell-Mann representation (λ_a) of the infinitesimal generators is

$$\begin{gathered}
T_1 = \frac{1}{2}\lambda_1 = \frac{1}{2}\begin{pmatrix} \sigma_1 & \vec{0}^T \\ \vec{0} & 0 \end{pmatrix}, \quad T_2 = \frac{1}{2}\lambda_2 = \frac{1}{2}\begin{pmatrix} \sigma_2 & \vec{0}^T \\ \vec{0} & 0 \end{pmatrix}, \quad T_3 = \frac{1}{2}\lambda_3 = \frac{1}{2}\begin{pmatrix} \sigma_3 & \vec{0}^T \\ \vec{0} & 0 \end{pmatrix} \\
V_1 = \frac{1}{2}\lambda_4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad V_2 = \frac{1}{2}\lambda_5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad V_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \\
U_1 = \frac{1}{2}\lambda_6 = \frac{1}{2}\begin{pmatrix} 0 & \vec{0} \\ \vec{0}^T & \sigma_1 \end{pmatrix}, \quad U_2 = \frac{1}{2}\lambda_7 = \frac{1}{2}\begin{pmatrix} 0 & \vec{0} \\ \vec{0}^T & \sigma_2 \end{pmatrix}, \quad U_3 = \frac{1}{2}\begin{pmatrix} 0 & \vec{0} \\ \vec{0}^T & \sigma_3 \end{pmatrix}, \\
Y = \frac{2}{\sqrt{3}}T_8 = \frac{2}{\sqrt{3}}\frac{1}{2}\lambda_8 = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}
\end{gathered} \tag{4.8}$$

▶ The generators $\frac{\lambda_i}{2}$ satisfy

$$\mathrm{tr}[\frac{\lambda_a}{2}\frac{\lambda_b}{2}] = \frac{1}{2}\delta_{ab}\,, \qquad \left[\frac{\lambda_a}{2}, \frac{\lambda_b}{2}\right] = i f_{abc}\frac{\lambda_c}{2} \tag{4.9}$$

with f_{abc} totally antisymmetric

$$f_{123} = 1,\; f_{147} = f_{246} = f_{257} = f_{345} = -f_{156} = -f_{367} = \frac{1}{2},\; f_{458} = f_{678} = \frac{\sqrt{3}}{2}\,.$$

Which can easily be proved by straight forward matrix multiplication.
▶ $\mathsf{SU}(3)$ is a **rank**-2 group, i.e. it has two simultaneously diagonal (hence commuting) generators T_3 and Y (the **Cartan subalgebra**). Or, in a basis independent

way, there are not more than 2 generators which mutually commute.

▶ In general, the **rank** of an algebra is the number of simultaneously commuting generators, and this set is called the **Cartan subalgebra**.

▶ We can write the $\mathfrak{su}(3)$ algebra in terms of 3 $\mathfrak{su}(2)$ subalgebras (not invariant, the algebra is simple)

$$T_\pm = T_1 \pm iT_2\,, \quad V_\pm = V_1 \pm iV_2 \quad \text{and} \quad U_\pm = U_1 \pm iU_2\,, \tag{4.10}$$

each set satisfying the familiar $SU(2)$ relations

$$\begin{aligned} &[T_3, T_\pm] = \pm\, T_\pm\,, && [T_+, T_-] = 2T_3 \\ &[U_3, U_\pm] = \pm\, U_\pm\,, && [U_+, U_-] = 2U_3 = \frac{3}{2}Y - T_3 \\ &[V_3, V_\pm] = \pm\, V_\pm\,, && [V_+, V_-] = 2V_3 = \frac{3}{2}Y + T_3 \end{aligned}$$

▶ Since the 3 subalgebras are linearly dependent, ($U_3 = -T_3 + V_3$), we also have relations between them

$$\begin{aligned} [T_3, U_\pm] = \mp\frac{1}{2}U_\pm\,, \quad [T_3, V_\pm] = \pm\frac{1}{2}V_\pm\,, \\ [Y, U_\pm] = \pm U_\pm, \quad [Y, V_\pm] = \pm V_\pm, \quad [Y, T_\pm] = 0 \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} &[T_+, V_+] = [T_+, U_-] = [U_+, V_+] = 0 && \text{(for angle, } 60^o\text{, see later)} \\ &[T_+, V_-] = -U_-, \quad [T_+, U_+] = V_+, \quad [U_+, V_-] = T_- && \text{(for angle, } 120^o\text{, see later)} \end{aligned} \tag{4.12}$$

In terms of Y, ($\sim$ hypercharge in hadron physics) we have

$$[Y, U_\pm] = \pm U_\pm, \quad [Y, V_\pm] = \pm V_\pm, \quad [Y, T_\pm] = 0. \tag{4.13}$$

Other commutation relations can be obtained from the ones listed here by hermitian conjugation, for example $([T_+, V_+])^\dagger$ gives the relation for $[T_-, V_-]$.

4.3.2 Step operators and states in $\mathfrak{su}(3)$

▶ For $\mathfrak{su}(2)$, the multiplets were labeled by the eigenvalues of the Casimir operator J^2 and the generator J_3. $\mathsf{SU}(2)$ is a rank 1 group, while $\mathsf{SU}(3)$ is rank 2 and has two independent Casimir operators. These, however, are typically not used to label the states. Instead, generally in the case of $\mathsf{SU}(3)$, a pair of numbers (p, q) is used to label the multiplets (cf. j rather than $j(j+1)$ for $SU(2)$).

► Just as we did in the case of $\mathfrak{su}(2)$, every state within a multiplet, of $\mathfrak{su}(3)$ can be seen an eigenstate of T_3. We could use the same thinking for the U_3 and V_3 generators.

► However, since these operators mutually commute, the state will be a simultaneous eigenstate of T_3, U_3 and V_3, i.e., we can label a state

$$|t_3, u_3, v_3\rangle\,, \tag{4.14}$$

where t_3, u_3 and v_3 are eigenvalues of the corresponding generators. This notation contains redundant information, but that is allowed.

► We can then write (using the simplified notation $d^{(p,q)}(T_i) \equiv \widehat{T}_i$)

$$\begin{aligned}
\widehat{T}_3 \,|t_3u_3v_3\rangle &= t_3 \,|t_3u_3v_3\rangle \\
\widehat{U}_3 \,|t_3u_3v_3\rangle &= u_3 \,|t_3u_3v_3\rangle \\
\widehat{V}_3 \,|t_3u_3v_3\rangle &= v_3 \,|t_3u_3v_3\rangle
\end{aligned}$$

► Let us now look at the action of the raising/lowering operators on the eigenstates:

★ Let's take the following state

$$\begin{aligned}
\widehat{T}_3 \left(\widehat{T_\pm}\,|t_3u_3v_3\rangle\right) &= \left(\widehat{T_\pm}\widehat{T}_3 + [\widehat{T}_3, \widehat{T_\pm}]\right)|t_3u_3v_3\rangle \\
&= \left(\widehat{T_\pm}\widehat{T}_3 \pm \widehat{T_\pm}\right)|t_3u_3v_3\rangle \\
&= (t_3 \pm 1)\left(\widehat{T_\pm}\,|t_3u_3v_3\rangle\right)
\end{aligned}$$

This tells us that $\widehat{T_\pm}\,|t_3u_3v_3\rangle$ is an eigenstate of $\widehat{T}_3$ with eigenvalue $t_3 \pm 1$, i.e. we can expand

$$\widehat{T_\pm}\,|t_3u_3v_3\rangle = \sum_{u_3'v_3'} N(t_3u_3v_3u_3'v_3')\,|t_3 \pm 1, u_3'v_3'\rangle$$

Note that u', v' may change.

★ Let us now consider the action of $\widehat{U_3}$ instead

$$\begin{aligned}
\widehat{U_3} \left(\widehat{T_\pm}\,|t_3u_3v_3\rangle\right) &= \left(\widehat{T_\pm}\widehat{U_3} + [\widehat{U_3}, \widehat{T_\pm}]\right)|t_3u_3v_3\rangle \\
&= \left(\widehat{T_\pm}\widehat{U_3} \mp \frac{1}{2}\widehat{T_\pm}\right)|t_3u_3v_3\rangle \\
&= \left(u_3 \mp \frac{1}{2}\right)\left(\widehat{T_\pm}\,|t_3u_3v_3\rangle\right)
\end{aligned}$$

Therefore, in a similar way as before, the above relation tells us that we can expand the state as

$$\widehat{T_\pm}\,|t_3u_3v_3\rangle = \sum_{t_3'v_3'} N(t_3u_3v_3t_3'v_3')\left|t_3', u_3 \mp \frac{1}{2}, v_3'\right\rangle$$

★ Lastly, we act with $\widehat{V}_3$

$$\begin{aligned}\widehat{V}_3\left(\widehat{T_\pm}\,|t_3u_3v_3\rangle\right) &= \left(\widehat{T_\pm}\widehat{V}_3 + [\widehat{V}_3, \widehat{T_\pm}]\right)|t_3u_3v_3\rangle \\ &= \left(\widehat{T_\pm}\widehat{V}_3 \pm \frac{1}{2}\widehat{T_\pm}\right)|t_3u_3v_3\rangle \\ &= \left(v_3 \pm \frac{1}{2}\right)\left(\widehat{T_\pm}\,|t_3u_3v_3\rangle\right)\end{aligned}$$

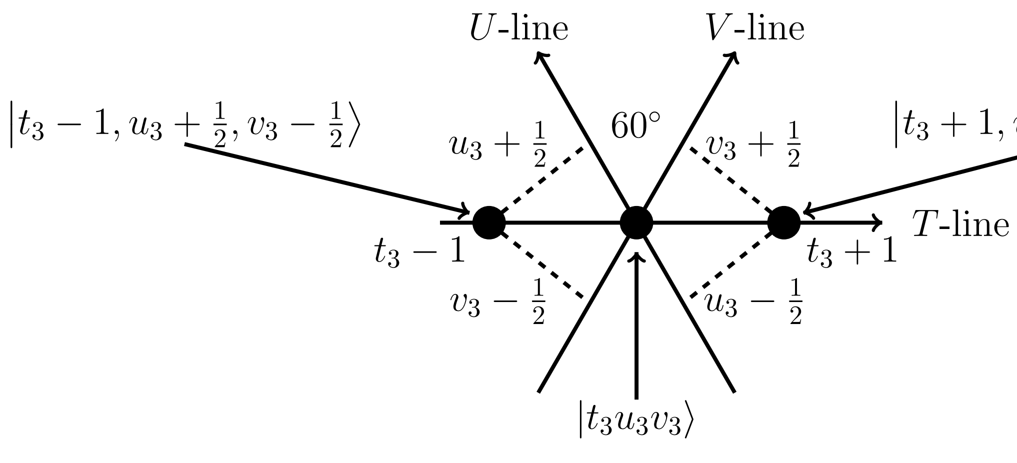
and therefore

$$\widehat{T_\pm}\,|t_3u_3v_3\rangle = \sum_{t_3'u_3'} N(t_3u_3v_3t_3'u_3')\left|t_3'u_3', v_3 \pm \frac{1}{2}\right\rangle$$

★ Collecting results, we can state the complete action of $T_\pm$ on the eigenstates

$$\widehat{T_\pm}\,|t_3u_3v_3\rangle \to \left|t_3 \pm 1, u_3 \mp \frac{1}{2}, v_3 \pm \frac{1}{2}\right\rangle$$

which can be represented graphically:



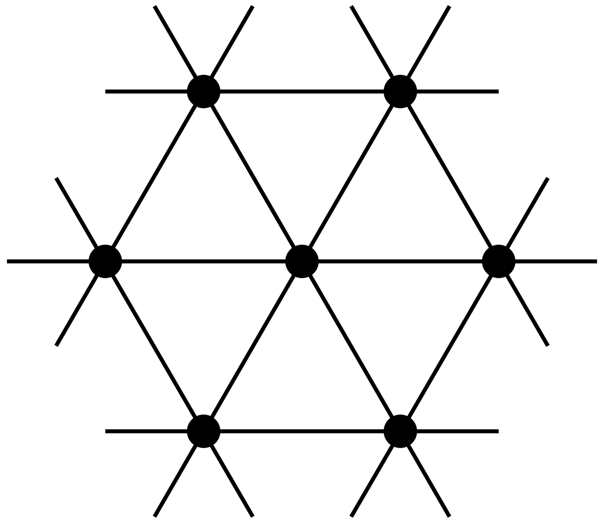
★ In a similar way, we can extract how the other raising/lowering operators $\widehat{U_\pm}/\widehat{V_\pm}$ act

$$\widehat{U_\pm}\,|t_3u_3v_3\rangle \to \left|t_3 \mp \frac{1}{2}, u_3 \pm 1, v_3 \pm \frac{1}{2}\right\rangle$$

$$\widehat{V_\pm}\,|t_3u_3v_3\rangle \to \left|t_3 \pm \frac{1}{2}, u_3 \pm \frac{1}{2}, v_3 \pm 1\right\rangle$$

4.3.3 General properties of multiplets in $\mathfrak{su}(3)$

▶ Repeated applications of the operators $T_\pm$, $U_\pm$ and $V_\pm$ generate a lattice of states



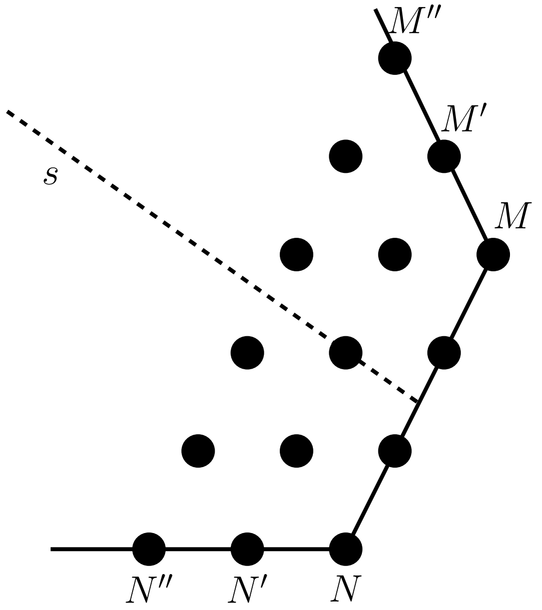
▶ When building $\mathfrak{su}(3)$ multiplets we can start from the state with the highest t_3-value, named the **highest weight** state and labeled $|\mathrm{max}\rangle$. This state satisfies

$$T_+ |\mathrm{max}\rangle = U_- |\mathrm{max}\rangle = V_+ |\mathrm{max}\rangle = 0 \,.$$

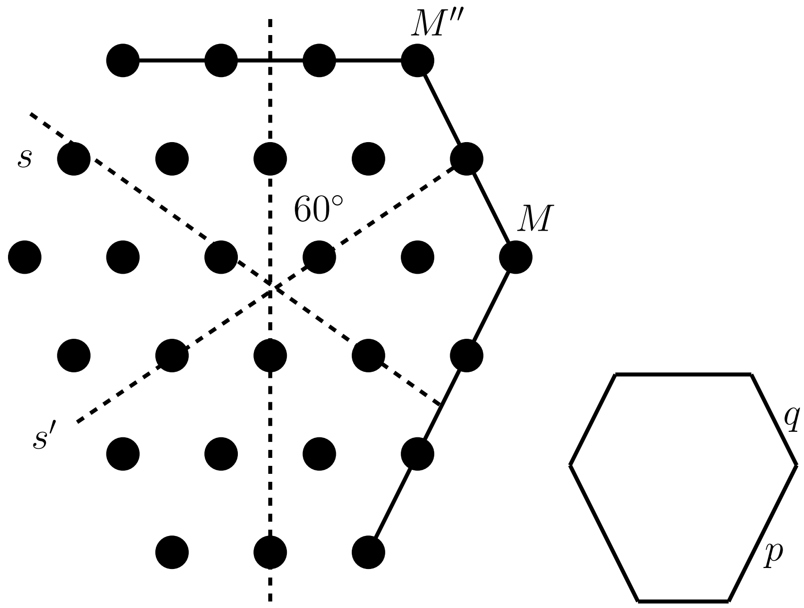
This is no unique way of defining a highest weight, we can find cases in the literature where $T_+ |\mathrm{max}\rangle = U_+ |\mathrm{max}\rangle = V_+ |\mathrm{max}\rangle = 0$, which would be the state with maximal u_3-value and largest t_3-value. We shall keep the first definition.

▶ The weight diagram has no concave angle. We will prove that in an exercise.

▶ Let us look at part of a multiplet. The point chain between M and N represents an $\mathfrak{su}(2)$-multiplet in V-direction. Correspondingly, the neighboring points of M must have a symmetrical counterpart in the neighborhood of N. That is, the $\mathfrak{su}(2)$-multiplet $N' - M'$ is also symmetric with respect to the perpendicular s-line. This is true for other V-multiplets as well, therefore, the U-line MM'' has the same length as the T-line N''.



► Now we start from the symmetry of the U-multiplet MM''. The remaining parallel submultiplets must also be symmetric with respect to the the perpendicular s' line. In this way the area of the $\mathfrak{su}(3)$-multiplet is closed forming a partially regular hexagon with three symmetry axes and angles of 60°.



► Letting p denote the length of the V-line going through the point with maximal weight, $p+1$ is the total number of state points on the V-line. Similarly, taking q to denote the length of the U-line passing through M, the number of states along that U-line is $q+1$. The integers p and q characterize the $\mathfrak{su}(3)$ multiplet completely, and we denote it by (p, q).

► Let us now look at the individual states of the (p, q)-multiplet:

★ First, let us characterize the state carrying maximal weight. The submultiplet on the V-line going through M has $p+1$ states. Because it is an $\mathfrak{su}(2)$-multiplet we have for the j-value along the V-line

$$\begin{array}{rcl} n \text{ states} & = & \dim R \\ p+1 & = & 2j_V+1 \end{array} \quad \Rightarrow \quad j_V = \frac{p}{2}\,.$$

★ In terms of the multiplet along the V-line, noting that the point M is the furthest member of this multiplet, $|M\rangle$ can be described as

$$|M\rangle = |j_V, m_{V,max}\rangle = \left|\frac{p}{2},\frac{p}{2}\right\rangle_V\,. \tag{4.15}$$

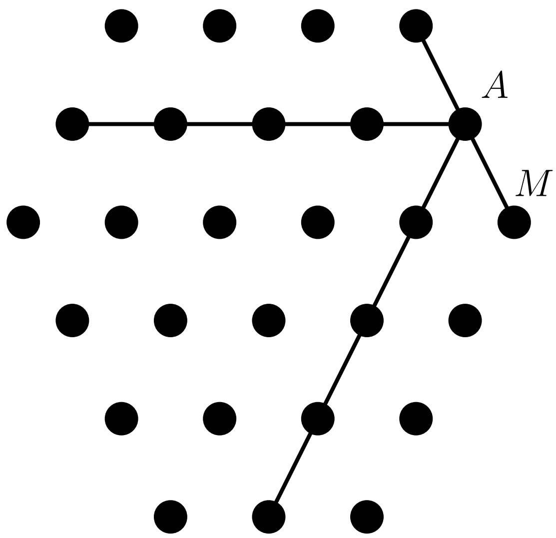
★ On the other hand, the state $|M\rangle$ is part of a U-multiplet

$$|M\rangle = \left|\frac{q}{2},-\frac{q}{2}\right\rangle_U \tag{4.16}$$

★ M is in the largest T-multiplet, and we will prove in exercise 4, problem sheet 2, that

$$|M\rangle = \left|\frac{p+q}{2},\frac{p+q}{2}\right\rangle_T\,.$$

★ In the above relations we have put the relative phases to +1. The state $|M\rangle$ is called unique, because it can be described by one expression of the $T-$, $U-$ or $V-$type.



► We will now show that the neighboring state $|A\rangle$ is also unique

★ $|A\rangle$ lies at the end of the V-multiplet with $p+2$ states, i.e.

$$|A\rangle = \left|\frac{p+1}{2},\frac{p+1}{2}\right\rangle_V \tag{4.17}$$

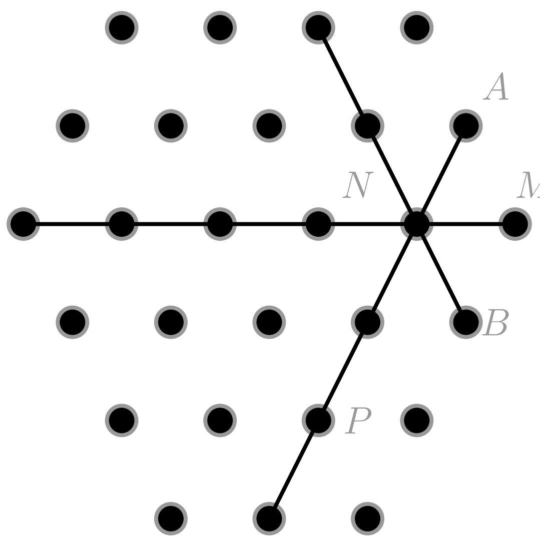
★ For the corresponding T-multiplet we have

$$|A\rangle = \left|\frac{p+q-1}{2}, \frac{p+q-1}{2}\right\rangle_T \tag{4.18}$$

★ In the U-subalgebra we get

$$|A\rangle = \left|\frac{q}{2}, -\frac{q}{2}+1\right\rangle_U \quad \text{and} \quad |A\rangle = \left|\frac{q}{2}-1, -\frac{q}{2}+1\right\rangle_U \tag{4.19}$$

★ **We define: a state is unique if it is represented at least in one subalgebra (T, U, V) by a single expression.** For this reason $|A\rangle$ is unique and we can drop the last expression.



▶ Another way to see this is by trying different paths from the highest weight sate $|M\rangle$ to states $|A\rangle$ or $|B\rangle$. For example, we can try the direct path to the states $|A, B\rangle$ and the one that first passes through the state $|M\rangle$, we get

$$\begin{cases} |A\rangle = \widehat{U_+}\,|M\rangle \\ |A'\rangle = \widehat{V_+}\widehat{T_-}\,|M\rangle = \left(\widehat{T_-}\widehat{V_+} - \widehat{U_+}\right)|M\rangle = -\widehat{U_+}\,|M\rangle \end{cases}$$

$$\begin{cases} |B\rangle = \widehat{V_-}\,|M\rangle \\ |B'\rangle = \widehat{U_-}\widehat{T_-}\,|M\rangle = \left(\widehat{T_-}\widehat{U_-} + \widehat{V_-}\right)|M\rangle = \widehat{V_-}\,|M\rangle \end{cases}$$

Up to a relative phase, $|A'\rangle$ and $|B'\rangle$ are the same states as $|A\rangle$ and $|B\rangle$, respectively. The same result is obtained for any other path.

▶ Now we deal with the state $|N\rangle$ and show that it is **doubly occupied**. This means that out of the 3 descriptions (T, U, V), one is composed out of a **pair of orthonormalized $\mathfrak{su}(2)$-states**. The other 2 descriptions may comprise two or more $\mathfrak{su}(2)$-states each. Therefore, we have:

★ We then get for the U-expressions

$$\left|\frac{q+1}{2},-\frac{q-1}{2}\right\rangle_U, \quad \left|\frac{q-1}{2},-\frac{q-1}{2}\right\rangle_U \tag{4.20}$$

which means that we can write the state $|N\rangle$ as a linear combination of these orthogonal expressions. The second expression is obtained by subtracting 1 to j, which accounts for deleting two sites. Essentially we extract the outer layer of sites.

★ For the T-subalgebra we get

$$\left|\frac{p+q}{2},\frac{p+q-2}{2}\right\rangle_T, \quad \left|\frac{p+q-2}{2},\frac{p+q-2}{2}\right\rangle_T \tag{4.21}$$

★ For the V-subalgebra we get

$$\left|\frac{p+1}{2},\frac{p-1}{2}\right\rangle_V, \quad \left|\frac{p-1}{2},\frac{p-1}{2}\right\rangle_V \tag{4.22}$$

★ The pairs of expressions are equivalent in such a way that the members of one pair, say T-pair, can be expressed linearly by the members of an other pair, say the U-pair

$$\begin{aligned}\left|\frac{p+q}{2},\frac{p+q-2}{2}\right\rangle_T =& a_1\left|\frac{q+1}{2},-\frac{q-1}{2}\right\rangle_U + a_2\left|\frac{q-1}{2},-\frac{q-1}{2}\right\rangle_U\\ \left|\frac{p+q-2}{2},\frac{p+q-2}{2}\right\rangle_T =& a_2\left|\frac{q+1}{2},-\frac{q-1}{2}\right\rangle_U - a_1\left|\frac{q-1}{2},-\frac{q-1}{2}\right\rangle_U\end{aligned}$$

The coefficients in the second expression have been chosen in order to have both combinations orthogonal.

★ In a similar way

$$\begin{aligned}\left|\frac{p+q}{2},\frac{p+q-2}{2}\right\rangle_T =& b_1\left|\frac{p+1}{2},\frac{p-1}{2}\right\rangle_V + b_2\left|\frac{p-1}{2},\frac{p-1}{2}\right\rangle_V\\ \left|\frac{p+q-2}{2},\frac{p+q-2}{2}\right\rangle_T =& b_2\left|\frac{p+1}{2},\frac{p-1}{2}\right\rangle_V - b_1\left|\frac{p-1}{2},\frac{p-1}{2}\right\rangle_V\end{aligned}$$

► In order to determine the coefficients we can proceed as follows:

★ We make use of the relation $U_+V_- - V_-U_+ - T_- = 0$ and apply it to the maximal weight state

$$\begin{aligned}0 =& (U_+V_- - V_-U_+ - T_-)\,|M\rangle\\ =& U_+V_-\left|\frac{p}{2},\frac{p}{2}\right\rangle_V - V_-U_+\left|\frac{q}{2},-\frac{q}{2}\right\rangle_U - T_-\left|\frac{p+q}{2},\frac{p+q}{2}\right\rangle_T\end{aligned}$$

★ The raising/lowering operators act just in the same way as in the case of $\mathfrak{su}(2)$-algebra. We then get

$$\begin{aligned}0 =& U_+\sqrt{\frac{p}{2}\frac{p+2}{2}-\frac{p}{2}\frac{p-2}{2}}\,|B\rangle - V_-\sqrt{\frac{q}{2}\frac{q+2}{2}-\frac{q}{2}\frac{q-2}{2}}\,|A\rangle \\ &- \sqrt{\frac{p+q}{2}\frac{p+q+2}{2}-\frac{p+q}{2}\frac{p+q-2}{2}}\left|\frac{p+q}{2},\frac{p+q-2}{2}\right\rangle_T \\ =& \sqrt{p}\sqrt{\frac{q+1}{2}\frac{q+3}{2}-\frac{q+1}{2}\frac{q-1}{2}}\left|\frac{q+1}{2},-\frac{q-1}{2}\right\rangle_U \\ &- \sqrt{q}\sqrt{\frac{p+1}{2}\frac{p+3}{2}-\frac{p+1}{2}\frac{p-1}{2}}\left|\frac{p+1}{2},\frac{p-1}{2}\right\rangle_V \\ &- \sqrt{p+q}\left|\frac{p+q}{2},\frac{q+p-2}{2}\right\rangle_T \\ =& \sqrt{p(q+1)}\left|\frac{q+1}{2},-\frac{q-1}{2}\right\rangle_U - \sqrt{q(p+1)}\left|\frac{p+1}{2},\frac{p-1}{2}\right\rangle_V \\ &- \sqrt{p+q}\left|\frac{p+q}{2},\frac{p+q-2}{2}\right\rangle_T\end{aligned}$$

★ We can now form the inner product with the bra state ${}_T\left\langle\frac{p+q}{2},\frac{p+q-2}{2}\right|$, i.e.

$$\begin{aligned}&\sqrt{p(q+1)}\quad {}_T\left\langle\frac{p+q}{2},\frac{p+q-2}{2}\right|\left.\frac{q+1}{2},-\frac{q-1}{2}\right\rangle_U \\ &-\sqrt{q(p+1)}\quad {}_T\left\langle\frac{p+q}{2},\frac{p+q-2}{2}\right|\left.\frac{p+1}{2},\frac{p-1}{2}\right\rangle_V \\ &-\sqrt{q+p}\quad {}_T\left\langle\frac{p+q}{2},\frac{p+q-2}{2}\right|\left.\frac{p+q}{2},\frac{p+q-2}{2}\right\rangle_T\end{aligned}$$

We then get

$$a_1\sqrt{p(q+1)} - b_1\sqrt{q(p+1)} - \sqrt{p+q} = 0$$

★ We can now form the inner product with the bra state ${}_T\left\langle\frac{p+q-2}{2},\frac{p+q-2}{2}\right|$ and get

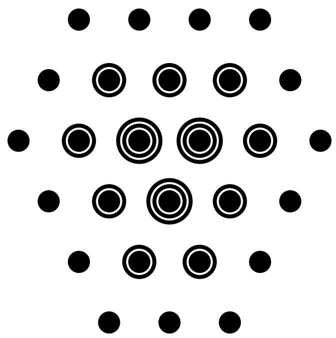
$$a_2\sqrt{p(q+1)} - b_2\sqrt{q(p+1)} = 0$$

★ Making use of the normalization $a_1^2+a_2^2=b_1^2+b_2^2=1$ we obtain

$$\begin{aligned}a_1 =& \sqrt{\frac{p}{(q+1)(p+q)}}, \quad a_2 = \sqrt{1-a_1^2} = \sqrt{\frac{q(p+q)+q}{(q+1)(p+q)}} \\ b_1 =& -\sqrt{\frac{q}{(q+1)(p+q)}}, \quad b_2 = \sqrt{\frac{q(p+q)+p}{(q+1)(p+q)}}\end{aligned}$$

▶ The states on the V-line between N and P contain more than two V-expressions. But since there are only two U- and T-expressions these states are said to be doubly occupied.

▶ Thus, **the states of the first inner shell are doubly occupied. The following shell is triply occupied** - provided that the preceding is hexagonal and so on.



▶ We can do the same by looking at the different paths that we have from $|M\rangle$ to $|N\rangle$. We have, for example,

$$\begin{aligned}
|N\rangle &= \widehat{T_-}\,|M\rangle \\
|N'\rangle &= \widehat{V_-}\widehat{U_+}\,|M\rangle = \widehat{U_+}\widehat{V_-}\,|M\rangle - \widehat{T_-}\,|M\rangle \\
|N''\rangle &= \widehat{U_+}\widehat{V_-}\,|M\rangle = |N'\rangle + |N\rangle
\end{aligned}$$

We see that $|N''\rangle$ can be written as a linear combination of $|N\rangle$ and $|N'\rangle$. The same will happen for any other path. As we go to an inner shell the number of independent paths increases by one unit.

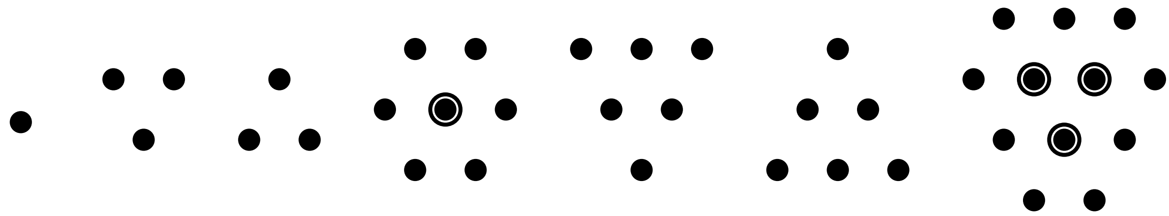
▶ The occupancy increases until we reach a triangular shell. After that, all triangular shells have the same occupancy.

▶ The dimension of the representation will be given by the sum of all weights, properly counting the multi occupied ones. The general expression for the dimension is given by

$$d(p,q) = \frac{1}{2}(p+1)(q+1)(p+q+2)\,.$$

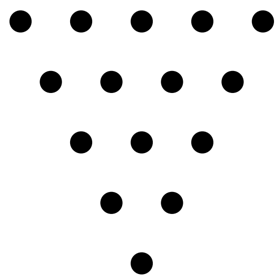
We will prove this relation in exercise 5 on the second exercise sheet. Note that this expression is symmetric in the interchange of p and q, i.e. the (p,q) and (q,p) representations have the same dimension.

▶ Here are some lower dimensional irreducible representations:



$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$	$(2,0)$	$(0,2)$	$(2,1)$
$\mathbf{1}$	$\mathbf{3}$	$\overline{\mathbf{3}}$	$\mathbf{8}$	$\mathbf{6}$	$\overline{\mathbf{6}}$	$\mathbf{15}$

► Note that we can have different irreducible representations with the same dimension. For example, we have see above that $(2,1)$ has dimension 15. The irreducible representation $(4,0)$ has the same dimension, but a different weight diagram



$(4,0)$ $\mathbf{15'}$

4.3.4 Irreducible representations of the $\mathfrak{su}(3)$ algebra

► We have that

$$\widehat{e}_i \left|\psi_k\right\rangle = \sum_{l=1}^{d} d^{(p,q)}(\widehat{e}_i)_{lk} \left|\psi_l\right\rangle = -\frac{i}{2}\sum_{l=1}^{d} d(\widehat{\lambda}_i)_{lk} \left|\psi_l\right\rangle$$

and, therefore, the matrices $-\frac{i}{2}D(\widehat{\lambda}_i)$ are representation of the Lie algebra $\mathfrak{su}(3)$. Therefore,

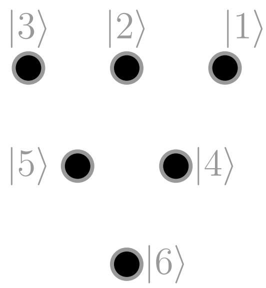
$$\widehat{\lambda}_i \left|\psi_k\right\rangle = \sum_{l=1}^{d} d^{(p,q)}(\widehat{\lambda}_i)_{lk} \left|\psi_l\right\rangle \quad \text{and} \quad D^{(p,q)}(\widehat{\lambda}_i)_{mk} = \left\langle \psi_m^{(p,q)} \middle| \widehat{\lambda}_i \middle| \psi_k^{(p,q)} \right\rangle$$

► We can write the operators $\widehat{\lambda}_i$ as

$$\begin{aligned}
\widehat{\lambda}_1 &= \widehat{T_+} + \widehat{T_-}, \quad \widehat{\lambda}_2 = -i\left(\widehat{T_+} - \widehat{T_-}\right), \quad \widehat{\lambda}_3 = 2\widehat{T}_3 \\
\widehat{\lambda}_4 &= \widehat{V_+} + \widehat{V_-}, \quad \widehat{\lambda}_5 = -i\left(\widehat{V_+} - \widehat{V_-}\right), \\
\widehat{\lambda}_6 &= \widehat{U_+} + \widehat{U_-}, \quad \widehat{\lambda}_7 = -i\left(\widehat{U_+} - \widehat{U_-}\right), \quad \widehat{\lambda}_8 = \frac{2}{\sqrt{3}}\left(\widehat{U}_3 + \widehat{V}_3\right)
\end{aligned}$$

► Let us look to an example, i.e. the $\mathfrak{su}(3)$ multiplet $(2,0)$:

★ The irrep $(2,0)$ is 6-plet, i.e. is characterized by the states $|1\rangle$, $|2\rangle$, . . . , $|6\rangle$



★ We now find the values of T_3, U_3 and V_3 that characterized the states. We start with the highest weight state

$$
\begin{aligned}
p+1 = 2j_V^{|1\rangle}+1 \quad &\longrightarrow \quad j_V^{|1\rangle} = \frac{p}{2} = 1 \rightarrow m_V^{|1\rangle} = 1,\, m_V^{|4\rangle} = 0,\, m_V^{|6\rangle} = -1 \\
q+1 = 2j_U^{|1\rangle}+1 \quad &\longrightarrow \quad j_U^{|1\rangle} = \frac{q}{2} = 0 \rightarrow m_U^{|1\rangle} = 0 \\
p+q+1 = 2j_T^{|1\rangle}+1 \quad &\longrightarrow \quad j_T^{|1\rangle} = \frac{p+q}{2} = 1 \rightarrow m_T^{|1\rangle} = 1,\, m_T^{|2\rangle} = 0,\, m_T^{|3\rangle} = -1
\end{aligned}
$$

For the state $|2\rangle$ we already have the T_3 value. Then

$$
\begin{aligned}
(p-1)+1 = 2j_V^{|2\rangle}+1 \quad &\longrightarrow \quad j_V^{|2\rangle} = \frac{p-1}{2} = \frac{1}{2} \rightarrow m_V^{|2\rangle} = \frac{1}{2},\, m_V^{|5\rangle} = -\frac{1}{2} \\
(q+1)+1 = 2j_U^{|2\rangle}+1 \quad &\longrightarrow \quad j_U^{|2\rangle} = \frac{q+1}{2} = \frac{1}{2} \rightarrow m_U^{|2\rangle} = \frac{1}{2},\, m_U^{|4\rangle} = -\frac{1}{2}
\end{aligned}
$$

Now we loot into the state $|3\rangle$

$$
\begin{aligned}
(p-2)+1 = 2j_V^{|3\rangle}+1 \quad &\longrightarrow \quad j_V^{|3\rangle} = \frac{p-2}{2} = 0 \rightarrow m_V^{|3\rangle} = 0 \\
(q+2)+1 = 2j_U^{|3\rangle}+1 \quad &\longrightarrow \quad j_U^{|3\rangle} = \frac{q+2}{2} = 1 \rightarrow m_U^{|3\rangle} = 1,\, m_U^{|5\rangle} = 0,\, m_U^{|6\rangle} = -1
\end{aligned}
$$

Now we loot into the state $|4\rangle$, we just need the T_3 value

$$
(p+q-1)+1 = 2j_T^{|4\rangle}+1 \quad \longrightarrow \quad j_T^{|4\rangle} = \frac{p+q-1}{2} = \frac{1}{2} \rightarrow m_T^{|4\rangle} = \frac{1}{2},\, m_T^{|5\rangle} = -\frac{1}{2}
$$

The only case left is the value of T_3 for $|6\rangle$, which is trivially zero. Therefore, we have final result

states	T_3	U_3	V_3
$\|1\rangle$	1	0	1
$\|2\rangle$	0	$\frac{1}{2}$	$\frac{1}{2}$
$\|3\rangle$	-1	1	0
$\|4\rangle$	$\frac{1}{2}$	$-\frac{1}{2}$	0
$\|5\rangle$	$-\frac{1}{2}$	0	$-\frac{1}{2}$
$\|6\rangle$	0	-1	-1

★ we now look at the action of $\widehat{T_\pm}$, $\widehat{U_\pm}$ and $\widehat{V_\pm}$. Since they will act of $\mathfrak{su}(2)$ triplets and doublet, we will have the factors $\sqrt{2}$ and 1 associated with each, respectively. For $\widehat{T_\pm}$ we have

$$
\begin{aligned}
\widehat{T_+}\,|1\rangle =&0\,, \quad \widehat{T_+}\,|2\rangle = \sqrt{2}\,|1\rangle\;, \quad \widehat{T_+}\,|3\rangle = \sqrt{2}\,|2\rangle \\
\widehat{T_-}\,|1\rangle =&\sqrt{2}\,|2\rangle\;, \quad \widehat{T_-}\,|2\rangle = \sqrt{2}\,|3\rangle\;, \quad \widehat{T_-}\,|3\rangle = 0 \\
\widehat{T_+}\,|4\rangle =&0\,, \quad \widehat{T_+}\,|5\rangle = |4\rangle \\
\widehat{T_-}\,|4\rangle =&|5\rangle\;, \quad \widehat{T_-}\,|5\rangle = 0 \\
\widehat{T_\pm}\,|6\rangle =&0
\end{aligned}
$$

For the $\widehat{U_\pm}$ we get

$$
\begin{aligned}
\widehat{U_\pm}\,|1\rangle =&0 \\
\widehat{U_+}\,|2\rangle =&0\,, \quad \widehat{U_+}\,|4\rangle = |2\rangle \\
\widehat{U_-}\,|2\rangle =&|4\rangle\;, \quad \widehat{U_-}\,|4\rangle = 0 \\
\widehat{U_+}\,|3\rangle =&0\,, \quad \widehat{U_+}\,|5\rangle = \sqrt{2}\,|3\rangle\;, \quad \widehat{U_+}\,|6\rangle = \sqrt{2}\,|5\rangle \\
\widehat{U_-}\,|3\rangle =&\sqrt{2}\,|5\rangle =, \quad \widehat{U_-}\,|5\rangle = \sqrt{2}\,|6\rangle\;, \quad \widehat{U_-}\,|6\rangle = 0
\end{aligned}
$$

and for $\widehat{V_\pm}$ we get

$$
\begin{aligned}
\widehat{V_+}\,|1\rangle =&0\,, \quad \widehat{V_+}\,|4\rangle = \sqrt{2}\,|1\rangle\;, \quad \widehat{V_+}\,|6\rangle = \sqrt{2}\,|4\rangle \\
\widehat{V_-}\,|1\rangle =&\sqrt{2}\,|4\rangle\;, \quad \widehat{V_-}\,|4\rangle = \sqrt{2}\,|6\rangle\;, \quad \widehat{V_-}\,|6\rangle = 0 \\
\widehat{V_+}\,|2\rangle =&0\,, \quad \widehat{V_+}\,|5\rangle = |2\rangle \\
\widehat{V_-}\,|2\rangle =&|5\rangle\;, \quad \widehat{V_-}\,|5\rangle = 0 \\
\widehat{V_\pm}\,|3\rangle =&0
\end{aligned}
$$

★ We can now compute the representation of the algebra generators

$$D(\widehat{\lambda}_1)^{(2,0)}_{mk} = \langle m| \left(\widehat{T_+} + \widehat{T_-}\right) |k\rangle = \langle m| \widehat{T_+} |k\rangle + \langle m| \widehat{T_-} |k\rangle$$

we then get

value	m	k	
$\sqrt{2}$	1	2	
$\sqrt{2}$	2	1	
$\sqrt{2}$	2	3	all other entries 0
$\sqrt{2}$	3	2	
1	4	5	
1	5	4	

The matrix then reads

$$D(\widehat{\lambda}_1)^{(2,0)} = \begin{pmatrix} 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

★ We can repeat the above procedure for the rest of the generators

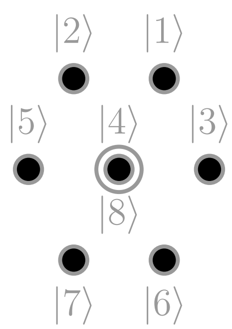
$$D(\widehat{\lambda}_2)^{(2,0)} = i\begin{pmatrix} 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \; D(\widehat{\lambda}_3)^{(2,0)} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$D(\widehat{\lambda}_4)^{(2,0)} = \begin{pmatrix} 0 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 \end{pmatrix}, \; D(\widehat{\lambda}_5)^{(2,0)} = i\begin{pmatrix} 0 & 0 & 0 & -\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 \end{pmatrix}$$

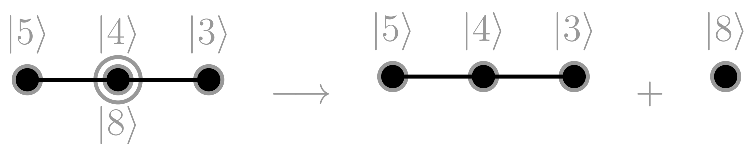
$$D(\widehat{\lambda}_6)^{(2,0)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 \end{pmatrix}, \; D(\widehat{\lambda}_7)^{(2,0)} = i \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{2} & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & -\sqrt{2} \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 \end{pmatrix}$$

$$D(\widehat{\lambda}_8)^{(2,0)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix}$$

► We can follow the same procedure in order to find the representations of other dimensional multiplets. In the case presented above there was no multi occupancy, let us then look at the $\mathfrak{su}(3)$ multiplet $(1, 1)$



In this case, when o look at the T multiplets. We have one double occupied weight diagram in $\mathfrak{su}(2)$, that can be decompose into its irreducible components



This is done in one of the $\mathfrak{su}(2)$ subalgebras, the other will be linear combinations of these states. (more details in problem 8 of week 2)

4.3.5 The hypercharge Y

► In order to specified the status of a $\mathfrak{su}(3)$-multiplet we used the T_3-, U_3- and V_3-values in a star-shaped coordinate system. Of course, two coordinates are sufficient for a two-dimensional system.

▶ In practice, the T-axis and a perpendicular Y-axis are taken. In hadron physics the Y-value are named hypercharge.

▶ The claim is that the hypercharge operator is given by

$$\widehat{Y} = \frac{1}{\sqrt{3}}\widehat{\lambda_8} = \frac{2}{3}\left(\widehat{U_3} + \widehat{V_3}\right) = \frac{2}{3}\left(2\widehat{U_3} + \widehat{T_3}\right) = \frac{2}{3}\left(2\widehat{V_3} - \widehat{T_3}\right)$$

★ Since $\widehat{Y}$ commutes with the TUV_3 operators, i.e. $[\widehat{Y}, \widehat{T_3}] = [\widehat{Y}, \widehat{U_3}] = [\widehat{Y}, \widehat{V_3}] = 0$, the state $|t_3 u_3 v_3\rangle$ is also eigenstate of the operator $\widehat{Y}$

$$\widehat{Y}\,|t_3 u_3 v_3\rangle = Y\,|t_3 u_3 v_3\rangle \quad \text{and} \quad |t_3 u_3 v_3\rangle \equiv |t_3 Y\rangle$$

★ Since $\widehat{Y}$ commutes with $\widehat{T_\pm}$, i.e.

$$[\widehat{Y}, \widehat{T_\pm}] = \frac{2}{3}[\widehat{U_3}, \widehat{T_\pm}] + \frac{2}{3}[\widehat{V_3}, \widehat{T_\pm}] = \frac{2}{3}\left(\mp\widehat{T_\pm} \pm \widehat{T_\pm}\right) = 0$$

leading to

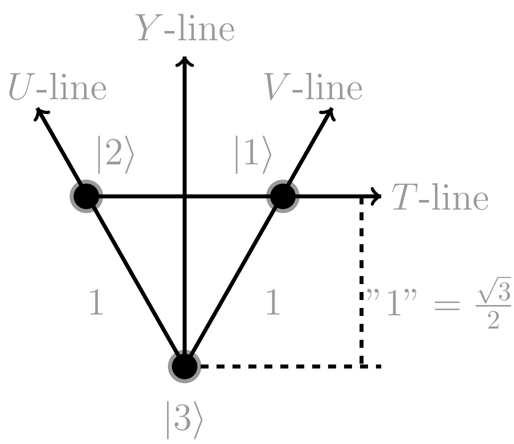
$$\widehat{Y}\left(\widehat{T_\pm}\,|t_3 Y\rangle\right) = Y\left(\widehat{T_\pm}\,|t_3 Y\rangle\right)$$

That is, the state $\widehat{T_\pm}\,|t_3 Y\rangle$, which differs in T_3 by ± 1 from the state $|t_3 Y\rangle$ has the same hypercharge. (No way, Sherlock!! That's why T_3- and Y-axis are orthogonal.)

★ Using $[\widehat{Y}, \widehat{U_\pm}] = \pm\widehat{U_\pm}$ we get

$$\begin{aligned}\widehat{Y}\left(\widehat{U_\pm}\,|t_3 Y\rangle\right) =& \widehat{U_\pm}\widehat{Y}\,|t_3 Y\rangle + \pm\widehat{U_\pm}\,|t_3 Y\rangle \\ =& (Y \pm 1)\left(\widehat{U_\pm}\,|t_3 Y\rangle\right)\end{aligned}$$

This means that $\widehat{U_\pm}$ not only rises and lowers U_3 by 1 but also Y in the same sense. Analogously on shows that Y varies alike to V_3. The picture below shows that behavior



The points 1 and 2 have both the distance 1 from 3. On the other hand, the distance (Y-value) of 3 from the T-line is also 1, which is geometrically impossible. We circumvent the inconsistency by taking the Y-values in units of $\sqrt{3}/2$.

Note: Reading the values of the hypercharge Y in units of $\sqrt{3}/2$ is nothing more than saying that the vertical axis is represented by $\widehat{T_8}$. We could have used the system T_8T_3 (which is the Cartan subalgebra), and then the we would have got

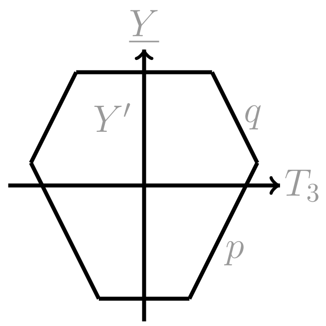
$$\widehat{T_8}\left(\widehat{U_\pm}\left|t_3t_8\right\rangle\right) = \left(t_8 \pm \frac{\sqrt{3}}{2}\right)\left(\widehat{U_\pm}\left|t_3t_8\right\rangle\right)$$

It is standard practice to use the hypercharge notation as it is related with physical quantities. But then in the weight diagrams we have to read it in units of $\sqrt{3}/2Y \equiv \underline{Y}$. I shall still call this the YT_3 plane but write $\underline{Y}$ to remember that this is not the true hypercharge.

▶ The origin of the TY-coordinate system lies on the intersection point of the symmetry axes s and s', because there the eigenvalues U_3 and V_3 vanish, i.e.

$$\widehat{Y}\left|t_3u_3v_3\right\rangle = \frac{2}{3}(\widehat{U_3} + \widehat{V_3})\left|t_3u_3v_3\right\rangle = 0$$

▶ The value of Y' is



$$Y' = \frac{\sqrt{3}}{2}\left(q + \frac{p-q}{3}\right)$$

Since p and q are integers, the expression $(p-q)/3$ amounts to an integer number plus 0, 1/3 or 2/3. Therefore, the points on top of the multiplet can have the Y-coordinates integer, integer+1/3 and integer+2/3 (measured in units of $\sqrt{3}/2$)

▶ This is true for all states pf a multiplet because they differ by integers if Y. One may say that the multiplet has the

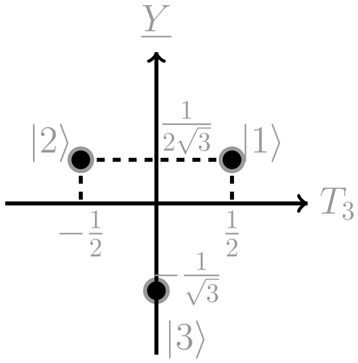
$$\text{Triality:} \quad \frac{\tau}{3} \quad (\tau = 1, 2, 3)$$

if its coordinates ate integer plus $\tau/3$.

4.3.6 Fundamental and antifundamental irreps weights

▶ We now look into more detail on the **3** and $\overline{\mathbf{3}}$ irreps. For the fundamental representation, i.e. **3** or $(1,0)$, we have

$\mathbf{3} = (1,0)$



$$\widehat{\underline{Y}}\,|1(2)\rangle = Y'\,|1(2)\rangle = \frac{1}{2\sqrt{3}}\,|1(2)\rangle$$

$$\begin{aligned}\widehat{\underline{Y}}\,|3\rangle = \widehat{\underline{Y}}\left(\widehat{U_-}\,|2\rangle\right) &= \left(\frac{1}{2\sqrt{3}} - 1.\frac{\sqrt{3}}{2}\right)|3\rangle \\ &= -\frac{1}{\sqrt{3}}\,|3\rangle\end{aligned}$$

▶ The weight vectors are then given by

State	Weight $(t_3, \overline{Y})$	Weight (t_3, Y)
$\vert 1\rangle$	$\left(\frac{1}{2}, \frac{1}{2\sqrt{3}}\right)$	$\left(\frac{1}{2}, \frac{1}{3}\right)$
$\vert 2\rangle$	$\left(-\frac{1}{2}, \frac{1}{2\sqrt{3}}\right)$	$\left(-\frac{1}{2}, \frac{1}{3}\right)$
$\vert 3\rangle$	$\left(0, -\frac{1}{\sqrt{3}}\right)$	$\left(0, -\frac{2}{3}\right)$

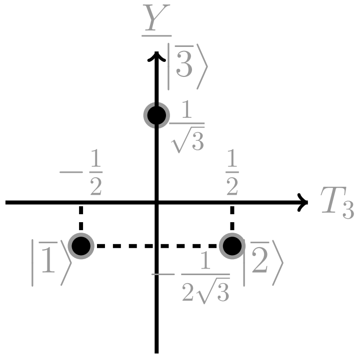
Thus the maximal weight state is $|1\rangle$ (largest T_3-value).

▶ Let us know look at the antifundamental representation, i.e. $\overline{\mathbf{3}}$ or $(0,1)$. Let us do it by steps:

- First we look at the maximal $\underline{Y}$-value, i.e Y'. For the representation $(0,1)$ is given by $Y' = \frac{1}{\sqrt{3}}$.
- Second we notice that since $p = 0$ in (p,q) this weight diagram has to be a triangle point up. This implies there is only one state with Y' value.

Therefore, we get

$\overline{\mathbf{3}} = (1, 0)$



$$\underline{\widehat{Y}}\,|\bar{3}\rangle = Y'\,|\bar{3}\rangle = \frac{1}{\sqrt{3}}\,|\bar{3}\rangle$$

$$\begin{aligned}\underline{\widehat{Y}}(-\,|\bar{2}\rangle) = \underline{\widehat{Y}}\left(\widehat{U_-}\,|\bar{3}\rangle\right) &= \left(\frac{1}{\sqrt{3}} - 1.\frac{\sqrt{3}}{2}\right)(-\,|\bar{2}\rangle) \\ &= \frac{-1}{2\sqrt{3}}(-\,|\bar{2}\rangle)\end{aligned}$$

(overall minus sign in $|2\rangle$ was omitted in the diagram, note that $\widehat{U_-}\,|\bar{3}\rangle = -\,|\bar{2}\rangle$.)

▶ The weight vectors are then given by

State	Weight $(t_3, \underline{Y})$	Weight (t_3, Y)
$\lvert\bar{1}\rangle$	$\left(-\frac{1}{2}, -\frac{1}{2\sqrt{3}}\right)$	$\left(-\frac{1}{2}, -\frac{1}{3}\right)$
$\lvert\bar{2}\rangle$	$\left(\frac{1}{2}, -\frac{1}{2\sqrt{3}}\right)$	$\left(\frac{1}{2}, -\frac{1}{3}\right)$
$\lvert\bar{3}\rangle$	$\left(0, \frac{1}{\sqrt{3}}\right)$	$\left(0, \frac{2}{3}\right)$

▶ We have found that if the fundamental states $|i\rangle$ have (t_3, Y) weight vectors, the states $|\bar{i}\rangle$ have $-(t_3, Y)$. This is actually a more general result: the weight vectors of the conjugated representation $\overline{\mathbf{R}}$ can be obtained from the weight vectors of $\mathbf{R}$ by a global sign conjugation.

▶ The reason why the weight vectors get a global minus sign can be easily understood by recalling how is defined the complex representation. We have seen in the previous chapter that

$$\text{if } T_a \text{ belongs to the algebra then } \overline{T_a} = -T_a^T \text{ also belongs.}$$

For anti-hermitian matrices the representation $d(\overline{T_a}) = d(T_a)^*$, i.e. the conjugated one. Let's keep working with the definition involving the transpose. We then have that the algebra operators satisfy

$$\begin{aligned}&\overline{\widehat{T_3}} = -\widehat{T_3}^{\,T} = -\widehat{T_3}\,, \quad \overline{\widehat{Y}} = -\widehat{Y}^T = -\widehat{Y} \quad \text{(diagonal operators)} \\ &\overline{\widehat{T_\pm}} = -\widehat{T_\pm}^{\,T} = -\widehat{T_\mp}\,, \quad \overline{\widehat{V_\pm}} = -\widehat{V_\pm}^{\,T} = -\widehat{V_\mp}\,, \quad \overline{\widehat{U_\pm}} = -\widehat{U_\pm}^{\,T} = -\widehat{U_\mp}\end{aligned}$$

▶ The fist line on the above equation tell us that the eigenvalues of $\widehat{T}_3$ and $\widehat{Y}$ change sign under conjugation. Just the result we wanted.

▶ The second line tell us something very important also. When raising and lowering states, we will get a minus sign factor in the action of the conjugate operators. Let us see this in action in the simplest example, we label $\widehat{T_\pm}^{\mathbf{3}} \equiv \widehat{T_\pm}$ and $\widehat{T_\pm}^{\overline{\mathbf{3}}} \equiv \overline{\widehat{T_\pm}}$.Take the states in the triplet and antitriplet irrep

$$\left|1(\overline{1})\right\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} , \quad \left|2(\overline{2})\right\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} , \quad \left|3(\overline{3})\right\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Diagrammatically we see that we can go from state $|1\rangle$ to $|2\rangle$ applying $\widehat{T_-}^{\mathbf{3}}$. In matrix form it would look like

$$\widehat{T_-}^{\mathbf{3}} |1\rangle = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = |2\rangle$$

For the complex representation we see that we should be able to go from $\left|\overline{1}\right\rangle$ to $\left|\overline{2}\right\rangle$ with the action $\widehat{T_+}^{\overline{\mathbf{3}}}$. In the matrix form this would look like

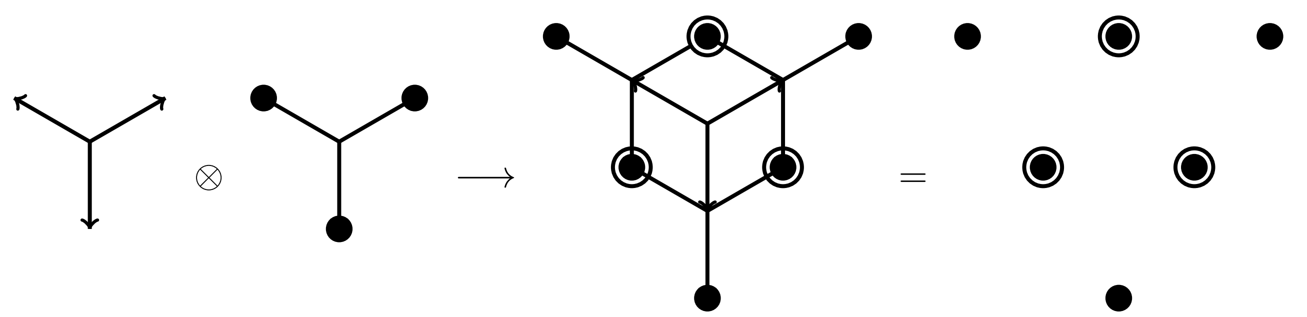
$$\widehat{T_+}^{\overline{\mathbf{3}}} \left|\overline{1}\right\rangle = -\widehat{T_-}^{\mathbf{3}} \left|\overline{1}\right\rangle = - \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = - \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = - \left|\overline{2}\right\rangle$$

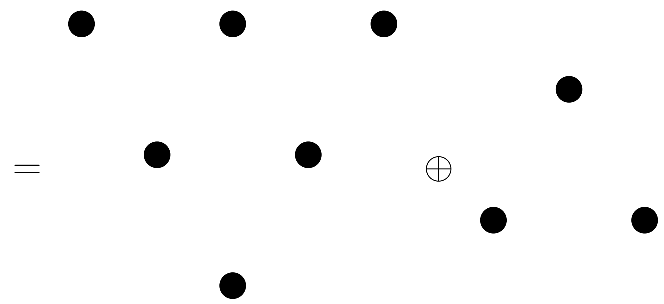
Therefore, the raising operator $\widehat{T_+}^{\overline{\mathbf{3}}}$ give us a new state with the T_3-value increased by one unit, but the an additional minus sight. When using the ladder operators for the conjugate representation we have to account for this extra sign.

4.3.7 Direct products of $\mathfrak{su}(3)$-multiplets

▶ We can extend the graphical procedure used in $\mathfrak{su}(2)$ to the $\mathfrak{su}(3)$ case. Again, we depict the function set of a direct product in such a way that we draw the first multiplet (p, q) and set the second multiplet (p', q') repeatedly so that its center appears in every point of (p, q), which we delete afterwards. For example:

▶ The tensor product $\mathbf{3} \otimes \mathbf{3}$

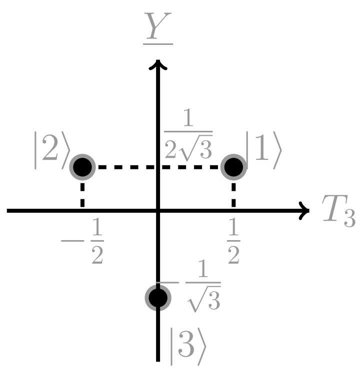




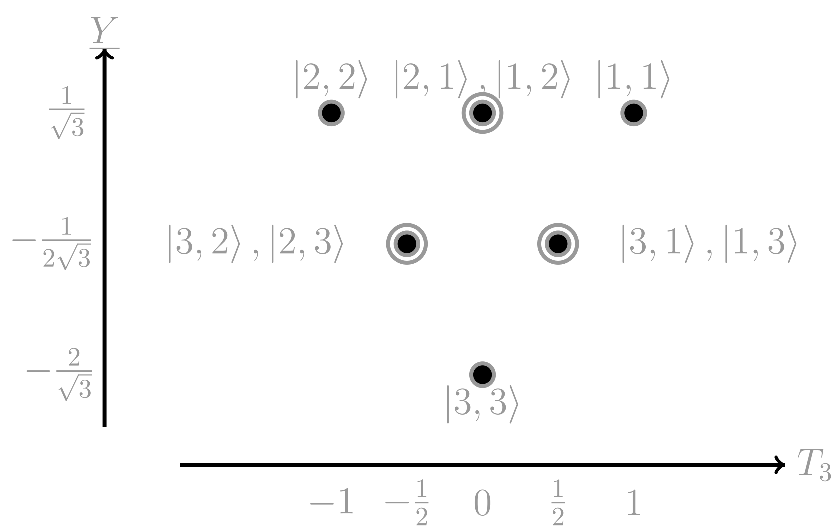
$$\left(\mathbf{3} \otimes \mathbf{3} = \mathbf{6} \oplus \overline{\mathbf{3}}\right)$$

The is the known Clebsch-Gordan series. ▶ We can label the multiplets and also look at the CG coefficients for the representations.

★ Let's say we have a triplet labeled as



★ We compute the product with itself



With $|i,j\rangle \equiv |i\rangle \otimes |j\rangle$. The eigenvalues can be computed by acting the diagonal operators on the final states

$$\underline{\widehat{Y}}\,|1,1\rangle = \left(\underline{\widehat{Y}}\,|1\rangle'\right) \otimes |1\rangle + |1\rangle \otimes \left(\underline{\widehat{Y}}\,|1\rangle'\right) = 2\frac{1}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$$
$$\underline{\widehat{Y}}\,|3,3\rangle = \left(\underline{\widehat{Y}}\,|3\rangle'\right) \otimes |3\rangle + |3\rangle \otimes \left(\underline{\widehat{Y}}\,|3\rangle'\right) = 2\frac{-1}{\sqrt{3}} = -\frac{2}{\sqrt{3}}$$

The weights on the middle line have in the Y-direction "1" unit less than the highest Y-value therefore $-\frac{1}{2\sqrt{3}}$. For the $\widehat{T}_3$ we have

$$\widehat{T}_3\,|1,1\rangle = \left(\widehat{T}_3\,|1\rangle'\right) \otimes |1\rangle + |1\rangle \otimes \left(\widehat{T}_3\,|1\rangle'\right) = 2\frac{1}{2} = 1\,.$$

This give the values of the first line of weights. For the second line, since is a doublet of $\mathfrak{su}(2)_T$ it has $\pm 1/2$.

★ We already know that the single states belong to the **6**. What we need to find is which combination of states enters in the **6** and $\bar{\mathbf{3}}$ for the doubly occupied ones. Therefore, we define

$$|1,2\rangle_{\mathbf{6}} = a\,|1\rangle \otimes |2\rangle + b\,|2\rangle \otimes |1\rangle\ , \quad |1,2\rangle_{\bar{\mathbf{3}}} = c\,|1\rangle \otimes |2\rangle + d\,|2\rangle \otimes |1\rangle\ .$$

★ The states should be orthogonal and normalized to unit, we then get the following constraints

$$a^2 + b^2 = 1\,, \quad c^2 + d^2 = 1\,, \quad ac + bd = 0\,.$$

Where we used the fact that $\mathfrak{su}(3)$ is a real Lie algebra.

★ We now act the raising operator $\widehat{T_+}$ on these states

$$\begin{aligned}
\widehat{T_+}^{\mathbf{6}}\,|1,2\rangle_{\mathbf{6}} =& a\left(\widehat{T_+}^{\mathbf{3}}\,|1\rangle \otimes |2\rangle + |1\rangle \otimes \widehat{T_+}^{\mathbf{3}}\,|2\rangle\right) \\
&+ b\left(\widehat{T_+}^{\mathbf{3}}\,|2\rangle \otimes |1\rangle + |2\rangle \otimes \widehat{T_+}^{\mathbf{3}}\,|1\rangle\right) \\
=&(a+b)\,|1\rangle \otimes |1\rangle \\
\widehat{T_+}^{\bar{\mathbf{3}}}\,|1,2\rangle_{\bar{\mathbf{3}}} =& c\left(\widehat{T_+}^{\mathbf{3}}\,|1\rangle \otimes |2\rangle + |1\rangle \otimes \widehat{T_+}^{\mathbf{3}}\,|2\rangle\right) \\
&+ d\left(\widehat{T_+}^{\mathbf{3}}\,|2\rangle \otimes |1\rangle + |2\rangle \otimes \widehat{T_+}^{\mathbf{3}}\,|1\rangle\right) \\
=&(c+d)\,|1\rangle \otimes |1\rangle
\end{aligned}$$

★ Since $\widehat{T_+}^{\bar{\mathbf{3}}}\,|1,2\rangle \equiv 0$ we get

$$d = -c \quad \Rightarrow \quad a = b = \frac{1}{\sqrt{2}}\,.$$

Leading to

$$\widehat{T_+}^{\mathbf{6}} |1,2\rangle_{\mathbf{6}} = \sqrt{2}\,|1,1\rangle_{\mathbf{6}}$$

like we would expect from a $SU(2)_T$ triplet.

★ The we get

$$|1,2\rangle_{\mathbf{6}} = \frac{1}{\sqrt{2}}\left(|1\rangle\otimes|2\rangle + |2\rangle\otimes|1\rangle\right), \quad |1,2\rangle_{\bar{\mathbf{3}}} = \frac{1}{\sqrt{2}}\left(-|1\rangle\otimes|2\rangle + |2\rangle\otimes|1\rangle\right)$$

★ We can do the same for $|2,3\rangle_{\mathbf{6},\bar{\mathbf{3}}}$ and $|1,3\rangle_{\mathbf{6},\bar{\mathbf{3}}}$ using $\widehat{U_+}$ and $\widehat{V_+}$, respectively. Let us do it for $|2,3\rangle_{\bar{\mathbf{3}}}$

$$\begin{aligned}\widehat{U_-}\,|2,1\rangle_{\bar{\mathbf{3}}} \equiv -\,|1,3\rangle_{\bar{\mathbf{3}}} =& \frac{\widehat{U_-}}{\sqrt{2}}\left(-|1\rangle\otimes|2\rangle + |2\rangle\otimes|1\rangle\right)\\ =& \frac{1}{\sqrt{2}}\left(-|1\rangle\otimes|3\rangle + |3\rangle\otimes|1\rangle\right)\end{aligned}$$

and

$$\begin{aligned}\widehat{V_-}\,|2,1\rangle_{\bar{\mathbf{3}}} \equiv -\,|2,3\rangle_{\bar{\mathbf{3}}} =& \frac{\widehat{V_-}}{\sqrt{2}}\left(-|1\rangle\otimes|2\rangle + |2\rangle\otimes|1\rangle\right)\\ =& \frac{1}{\sqrt{2}}\left(-|3\rangle\otimes|2\rangle + |2\rangle\otimes|3\rangle\right)\end{aligned}$$

As a self-consistency test we can check that

$$\widehat{T_+}(-\,|2,3\rangle_{\bar{\mathbf{3}}}) = -\,|1,3\rangle_{\bar{\mathbf{3}}}$$

where we have used $\widehat{T_+} = [\widehat{V_+}, \widehat{U_-}]$;

★ We then get

$$
\begin{array}{ll|l|l}
 & \text{states} & \text{weight } (t_3, \underline{Y}) & \text{weight } (t_3, Y) \\
\mathbf{6}: & |1\rangle \otimes |1\rangle & \left(1, \frac{1}{\sqrt{3}}\right) & \left(1, \frac{2}{3}\right) \\
 & |2\rangle \otimes |2\rangle & \left(-1, \frac{1}{\sqrt{3}}\right) & \left(-1, \frac{2}{3}\right) \\
 & |3\rangle \otimes |3\rangle & \left(0, -\frac{2}{\sqrt{3}}\right) & \left(0, -\frac{4}{3}\right) \\
 & \frac{1}{\sqrt{2}}\left(|1\rangle \otimes |2\rangle + |2\rangle \otimes |1\rangle\right) & \left(0, \frac{1}{\sqrt{3}}\right) & \left(0, \frac{2}{3}\right) \\
 & \frac{1}{\sqrt{2}}\left(|1\rangle \otimes |3\rangle + |3\rangle \otimes |1\rangle\right) & \left(\frac{1}{2}, -\frac{1}{2\sqrt{3}}\right) & \left(\frac{1}{2}, -\frac{1}{3}\right) \\
 & \frac{1}{\sqrt{2}}\left(|2\rangle \otimes |3\rangle + |3\rangle \otimes |2\rangle\right) & \left(-\frac{1}{2}, -\frac{1}{2\sqrt{3}}\right) & \left(-\frac{1}{2}, -\frac{1}{3}\right) \\
\bar{\mathbf{3}}: & \frac{1}{\sqrt{2}}\left(-|1\rangle \otimes |2\rangle + |2\rangle \otimes |1\rangle\right) & \left(0, \frac{1}{\sqrt{3}}\right) & \left(0, \frac{2}{3}\right) \\
 & \frac{1}{\sqrt{2}}\left(-|1\rangle \otimes |3\rangle + |3\rangle \otimes |1\rangle\right) & \left(\frac{1}{2}, -\frac{1}{2\sqrt{3}}\right) & \left(\frac{1}{2}, -\frac{1}{3}\right) \\
 & \frac{1}{\sqrt{2}}\left(-|3\rangle \otimes |2\rangle + |2\rangle \otimes |3\rangle\right) & \left(-\frac{1}{2}, -\frac{1}{2\sqrt{3}}\right) & \left(-\frac{1}{2}, -\frac{1}{3}\right)
\end{array}
$$

Therefore, **6** is symmetric and $\bar{\mathbf{3}}$ is antisymmetric under the permutation of two of the original states.

4.4 A short note on the general classification of complex semi-simple Lie algebras

Read Jones 9.2-9.5 instead of this section.

▶ Let's say that a particular Lie group has N generators total, or is a N dimensional group. Then, let's say that there are $M < N$ generators in the commuting subalgebra.

$$
\begin{aligned}
\text{Cartan Subalgebra:} &\quad \widehat{H}_i \, (i = 1, \cdots, M) \\
\text{non-Cartan generators:} &\quad \widehat{E}_i \, (i = 1, \cdots, N - M) \\
\text{Rank of the group:} &\quad M
\end{aligned}
$$

and $\widehat{H}_i$ hermitian.

★ $\widehat{H}_i$ are by definition simultaneously diagonalized. We will write physical states in terms of their eigenvalues.

★ In an n-dimensional representation $d^{(n)}$, the generators are $n \times n$ matrices, there will be a total of n eigenvectors, and each will have one eigenvalue at each of the M Cartan generators H^i.

★ For each of these eigenvectors we have the M eigenvalues of the Cartan generators, which we call

$$\text{Weights: } \mu_j^i \to \begin{cases} j = 1, \cdots, n \text{ eigenvectors} \\ i = 1, \cdots, M \text{ eigenvalues} \end{cases} \qquad \text{Weight Vector: } \vec{\mu}_j = \begin{pmatrix} \mu_j^1 \\ \mu_j^2 \\ \mu_j^3 \\ \vdots \\ \mu_j^M \end{pmatrix}$$

with μ_j^i real parameters, because these are eigenvalues of hermitian operators.

▶ The remaining non-Cartan generators can be combined into a linear combinations to form a linearly independent set of raising and lowering operators. These are also called Weyl generators.

▶ Weight vectors for the adjoint representation are called roots (or root vectors) and are the objects that tell us in which direction we can move and how much in the M-dimensional space.

▶ In the Cartan-Weyl basis the algebra reads

$$[H^i, H^j] = 0\,, \quad [H^i, E^{\vec{\alpha}}] = \alpha_i E^{\vec{\alpha}}\,, \quad [E^{\vec{\alpha}}, E^{-\vec{\alpha}}] = \alpha_i H^i\,,$$
$$[E^{\vec{\alpha}}, E^{\vec{\beta}}] = \begin{cases} N_{\alpha\beta} E^{\vec{\alpha}+\vec{\beta}} & \vec{\alpha} + \vec{\beta} \neq 0 \\ 0 \end{cases}$$

Nonzero roots are non-degenerate and roots are not proportional apart from $\vec{\alpha}$ and $-\vec{\alpha}$.

▶ For semi-simple Lie algebras we have some rules for the roots:

★ If $\vec{\alpha}$ is a root, so is $-\vec{\alpha}$;

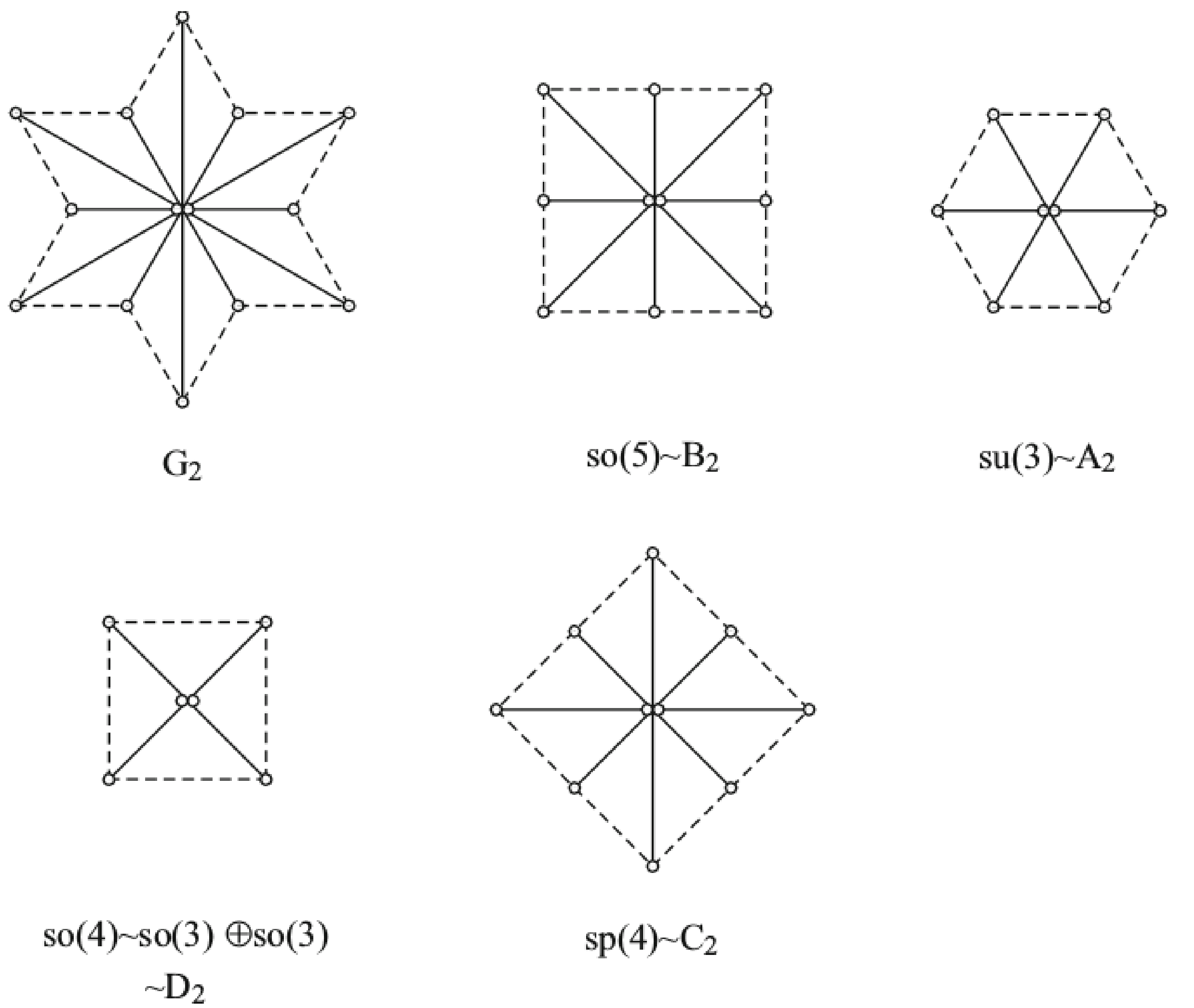
★ If $\vec{\alpha}$, $\vec{\beta}$ are roots, $\frac{2(\vec{\alpha},\vec{\beta})}{(\vec{\alpha},\vec{\beta})}$ is an integer;

★ If $\vec{\alpha}$, $\vec{\beta}$ are roots, $\beta - 2\vec{\alpha}\frac{(\vec{\alpha},\vec{\beta})}{(\vec{\alpha},\vec{\alpha})}$ is a root.

▶ From this it follows that the angle φ between roots is given by

$$\cos\varphi = \frac{(\vec{\alpha},\vec{\beta})}{\sqrt{(\vec{\alpha},\vec{\alpha})(\vec{\beta},\vec{\beta})}} \quad \longrightarrow \quad \varphi = 30^\circ,\ 45^\circ,\ 60^\circ,\ 90^\circ\,.$$

▶ The root diagrams in 2D are given below



▶ Let us then look back to what we did in $\mathfrak{su}(3)$.

★ Looking at the 3-dimensional representation, we will have 3 eigenvectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} , \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} , \quad \vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{4.23}$$

★ The Cartan generators

$$H^1 = T_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H^2 = T_8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{4.24}$$

★ The eigenvalues are

$$\begin{aligned} H^1\vec{v}_1 &= \left(\frac{1}{2}\right)\vec{v}_1\,, \quad H^1\vec{v}_2 = \left(-\frac{1}{2}\right)\vec{v}_2\,, \quad H^1\vec{v}_3 = (0)\,\vec{v}_3\,, \\ H^2\vec{v}_1 &= \left(\frac{1}{2\sqrt{3}}\right)\vec{v}_1\,, \quad H^2\vec{v}_2 = \left(\frac{1}{2\sqrt{3}}\right)\vec{v}_2\,, \quad H^2\vec{v}_3 = \left(-\frac{1}{\sqrt{3}}\right)\vec{v}_3 \end{aligned}$$

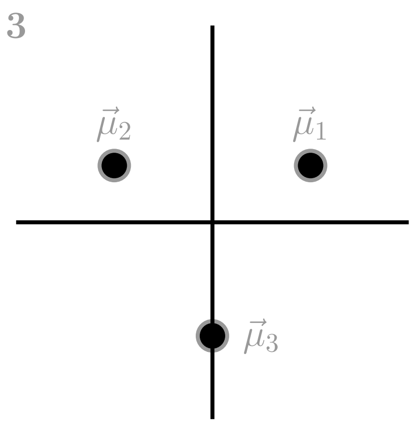
★ So the eigenvector $\vec{v}_1$ has eigenvalue $1/2$ with H^1 and eigenvalue $1/2\sqrt{3}$ with H^2. This is consistent with the general fact that the dimension of the root space is always equal to the rank $(n-1 = 3-1 = 2)$.

So the weight vectors are

$$\vec{\mu}_1 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2\sqrt{3}} \end{pmatrix}, \quad \vec{\mu}_2 = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2\sqrt{3}} \end{pmatrix}, \quad \vec{\mu}_3 = \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{3}} \end{pmatrix} \tag{4.25}$$

they **are not** called "root" vectors because we are not working in the adjoint representation.

★ We can draw a graph in the 2-dimensional (H^1, H^2) plane as below



► So, now we have a good understanding of the "charge lattice" in this representation of $SU(3)$, which depended only on the Cartan generators H^1 and H^2.

Since $\vec{t}_i$ are directly related with $\vec{v}_i$, the above relations implies

$$\begin{array}{llll} T^{\pm}: & \vec{\mu}_{2,1} \to \vec{\mu}_{1,2} & \text{meaning} & \vec{\mu}_2 + \vec{\alpha}_1 = \vec{\mu}_1 \\ V^{\pm}: & \vec{\mu}_{3,1} \to \vec{\mu}_{1,3} & \text{meaning} & \vec{\mu}_3 + \vec{\alpha}_2 = \vec{\mu}_1 \\ U^{\pm}: & \vec{\mu}_{3,2} \to \vec{\mu}_{2,3} & \text{meaning} & \vec{\mu}_3 + \vec{\alpha}_3 = \vec{\mu}_2 \end{array}$$

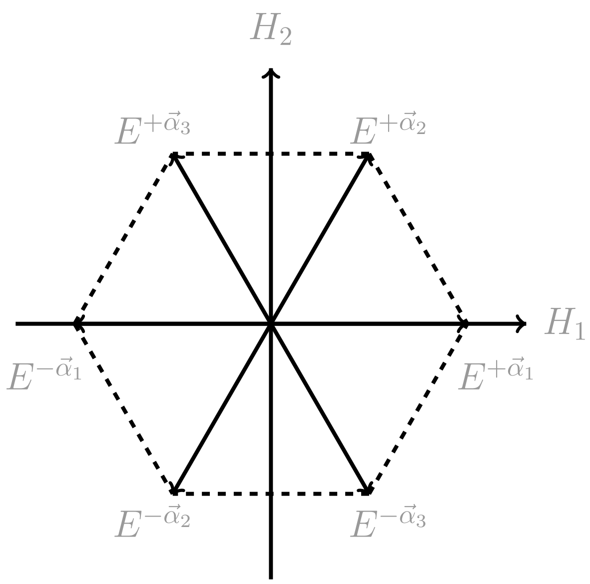
The solutions for $\vec{\alpha}_i$ are then given by

$$\vec{\alpha}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \quad \vec{\alpha}_2 = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} , \quad \vec{\alpha}_3 = \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}$$

These are **roots**, i.e. the weighs of the adjoint representation, since their parametrize the steps we are allowed to make in the weight diagram.

▶ The **roots** $\vec{\alpha}_i$ are not linear independent. We can choose two the build the linear independent space. We shall choose $\vec{\alpha}_2$ and $-\vec{\alpha}_3$, this choice is called **simple roots**, i.e. we have choose to work with the ladder operators

$$\vec{\alpha}_2 \sim V^+ \equiv E^{+\vec{\alpha}_2} \quad \text{and} \quad -\vec{\alpha}_3 \sim U^- \equiv E^{-\vec{\alpha}_3} \tag{4.26}$$



▶ In this way, all weight vectors are related

$$\vec{\mu}' = \vec{\mu} + l\vec{\alpha}_2 + m\vec{\alpha}_3 \tag{4.27}$$

with l and m integers.

▶ Higher dimensional representations are constructed diagrammatically using the weight and root vectors. To do this, we need to order and define larger or smaller vectors under an appropriate rule. Compare the first components of two vectors

$\vec{\mu}$ and $\vec{\nu}$. If $\mu_1 > \nu_1$, we define $\vec{\mu} > \vec{\nu}$. But if $\mu_1 = \nu_1$, we compare the second components. For instance

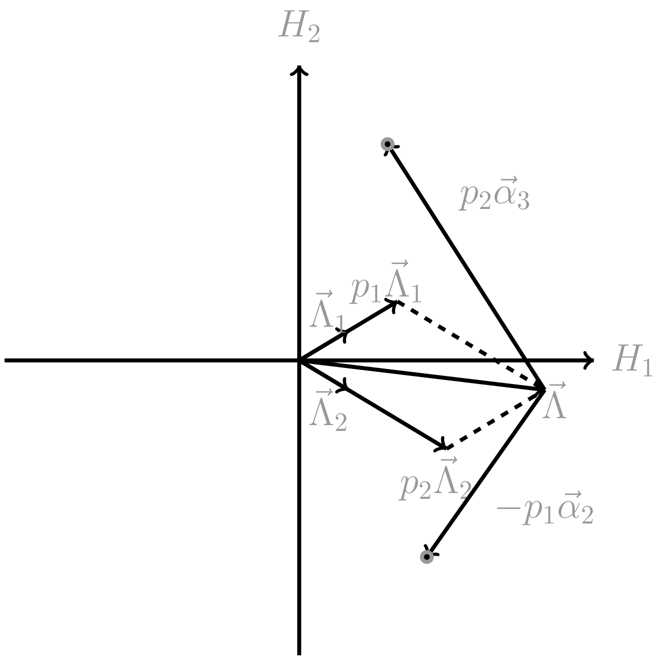
$$(3,1) > (2,3) \quad (2,1) > (2,0)\,. \tag{4.28}$$

► Now we can introduce the **highest weight** for each representation. For the (anti-) fundamental representations we get

$$\left.\begin{aligned} \mathbf{3} = (1,0):\ \vec{\Lambda}_1 \equiv \vec{\mu}_1 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2\sqrt{3}} \end{pmatrix} \\ \\ \overline{\mathbf{3}} = (0,1):\ \vec{\Lambda}_2 \equiv \vec{\bar{\mu}}_2 = \begin{pmatrix} \frac{1}{2} \\ \frac{-1}{2\sqrt{3}} \end{pmatrix} \end{aligned}\right\} \quad \begin{array}{l} \vec{\Lambda} = p_1\vec{\Lambda}_1 + p_2\vec{\Lambda}_2 \\ \text{highest weight for arbitrary irrep} \end{array}$$

where $p_{1,2}$ are non-negative integers.

Starting from the highest weight $\vec{\Lambda}$, application of $-\vec{\alpha}_2$ by p_1 times (subtracting to the highest weight the root $\vec{\alpha}_2$) determines one side $-p_1\vec{\alpha}_2$. Similarly one has another side $p_2\vec{\alpha}_3$ (subtraction to the highest weight the root $-\vec{\alpha}_3$)



Fundamental representation $(1,0) = \mathbf{3}$

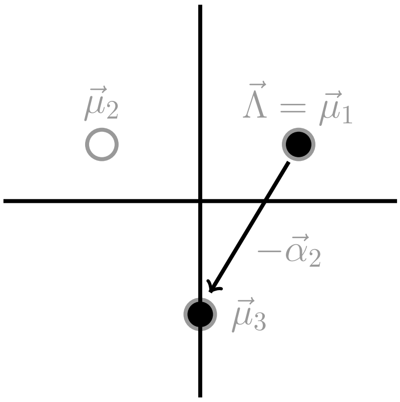
highest weight: $\vec{\Lambda} = 1.\vec{\Lambda}_1 + 0.\vec{\Lambda}_2 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2\sqrt{3}} \end{pmatrix} = \vec{\mu}_1$

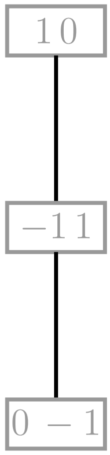
$\downarrow$

applying $-\vec{\alpha}_2$: $\vec{\Lambda} - \vec{\alpha}_2 = \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{3}} \end{pmatrix} = \vec{\mu}_3$

$\downarrow$

applying $+\vec{\alpha}_3$: $\vec{\mu}_3 + \vec{\alpha}_3 = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2\sqrt{3}} \end{pmatrix} = \vec{\mu}_2$





Chapter 5

From small boxes comes great responsibility!

Young diagrams and tensor formalism in $SU(N)$

5.1 Young diagrams

5.1.1 Connection with the permutation group

▶ The diagrammatic method presented in the previous section starts getting to over whelming for large (p, q) multiplets in $\mathfrak{su}(3)$, or larger $\mathfrak{su}(N)$. There is a more efficient method known as the Young diagrams that can help us in this task.

▶ In order to be able to work with Young diagrams we first need to introduce the idea of **partitions**:

(i) A partition $\lambda := \{\lambda_1, \lambda_2, \cdots, \lambda_r\}$ of the integer n is a sequence of positive integers λ_i, arranged in descending order, whose sum is equal to n

$$\boxed{\lambda_i \geq \lambda_{i+1}\,, \quad i = 1, \cdots, r-1} \quad \text{and} \quad \boxed{\sum_{i=1}^{r} \lambda_i = n}$$

(ii) Two partitions λ, μ are equal if $\lambda_i = \mu_i \,\forall_i$.

(iii) $\lambda > \mu$ ($\lambda < \mu$) if the first non-zero number in the sequence $(\lambda_i - \mu_i)$ is positive (negative).

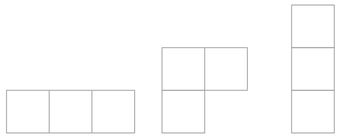
(iv) A partition λ is represented graphically by a **Young Diagram** which consists of n squares arranged in r rows, the ith one of which contains λ_i squares.

▶ **It's time for a couple of examples:**

★ **Take the case** $n = 3$, there are three distinct partitions and corresponding Young diagrams

Partition [3] [21] [111]

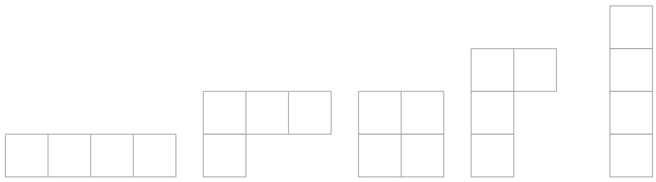
Young diag.



★ **Take the case of** $n = 4$, there are five distinct partitions and corresponding Young diagrams

Partition [4] [31] [22] [211] [1111]

Young diag.



Why is this useful? There is a one-to-one correspondence between the partitions of n and the classes of group elements S_n. Recalling what we have learn in the discrete part of the course, every class of S_n is characterized by a given **cycle structure** consisting of ν_1 1-cycles, ν_2 2-cycles, $\cdots$ etc. Since the numbers $1, 2, \cdots, n$ fill all the cycles, we must have

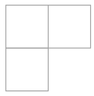
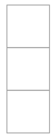
$$n = \nu_1 + 2\nu_2 + 3\nu_3 + \cdots = \underbrace{(\nu_1 + \nu_2 + \cdots)}_{\lambda_1} \underset{>}{+} \underbrace{(\nu_2 + \nu_3 + \cdots)}_{\lambda_2} \underset{>}{+} \underbrace{(\nu_3 + \cdots)}_{\lambda_3} \underset{>}{+} \cdots$$

and thus $\lambda := \{\lambda_i\}$ is a partition of n. This then lead us to the following theorem

Theorem

The number of distinct Young diagrams for any given n is equal to the number of classes of S_n – which is, in turn, equal to the number of inequivalent irreducible representations of S_n

Let us look at some examples. **For** S_3 we have the following classes and corresponding partitions and Young diagrams

Class	$\{e\}$	$\{(12),(23),(31)\}$	$\{(123),(321)\}$
Cycles struct.	$\nu_1 = 3,\ \nu_2 = \nu_3 = 0$	$\nu_1 = \nu_2 = 1,\ \nu_3 = 0$	$\nu_1 = \nu_2 = 0,\ \nu_3 = 1$
$(\lambda_1, \lambda_2, \lambda_3)$	$(3,0,0)$	$(2,1,0)$	$(1,1,1)$
Partition	[3]	[21]	[111]
Young diag.			

▶ In the following we define the permutations mathematically. For this purpose we imagine **non-identical objects** which are placed in the individual **identical** boxes. Then we denote the a permutation by

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & N \\ b_1 & b_2 & b_3 & \cdots & b_N \end{pmatrix}$$

meaning that the object which initially was in the first box has now been moved into box b_1, the one in the second box is transferred to b_2, etc.

▶ There are several different ways to express the distribution of objects into boxes or that of particles into states (equal particles different quantum states). Let α, β, γ, δ be four nonidentical objects or four nonidentical one-particle quantum states (wave-functions). Then

$$\alpha_1 \beta_2 \gamma_3 \delta_4$$

means that

QM interpretation

$$\left\{ \begin{array}{l} \text{object } \alpha \text{ is situated in box 1} \\ \text{object } \beta \text{ is situated in box 2} \\ \text{object } \gamma \text{ is situated in box 3} \\ \text{object } \delta \text{ is situated in box 4} \end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{l} \text{particle 1 is in state } \alpha \\ \text{particle 2 is in state } \beta \\ \text{particle 3 is in state } \gamma \\ \text{particle 4 is in state } \delta \end{array} \right.$$

We shall use the QM interpretation. In this interpretation, each box is a particle and the labeling is the state in which it is in.

▶ Let us look at one example

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \alpha_1 \beta_2 \gamma_3 \delta_4 = \alpha_2 \beta_3 \gamma_4 \delta_1$$

Now particle 2 is in state α, and so on. We can simplify this notation even more if we agree that the state of particle 1 is always noted first, followed by the state of particle 2 and so on. In this way the above transformation reads

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \alpha\beta\gamma\delta = \delta\alpha\beta\gamma\,.$$

▶ Sometimes the state is simply written as $\psi(1,2,3,4)$ instead of $\alpha_1\beta_2\gamma_3\delta_4$. This is more general in so far as $\psi(1,2,3,4)$ need not necessarily be a product of one-particle states $(\alpha\beta\gamma\delta)$. In this notation the correspondence to $\psi(2,3,4,1)$ is

$$\psi(2,3,4,1) \quad \Leftrightarrow \quad \alpha_2\beta_3\gamma_4\delta_1 \quad \text{or} \quad \delta\alpha\beta\gamma\,.$$

▶ As an example let us consider for simplicity states of 2 identical particles (S_2) with the two-particle state $\psi(1,2)$. The numbers 1 and 2 comprise all coordinates (position, spin, isospin) of the particles 1 and 2 respectively. In order to examine the symmetry under exchange of particles, we note that in general $\psi(1,2)$ does not have any particular symmetry. But we can always build a symmetric (ψ_s) and an antisymmetric (ψ_a) state out of $\psi(1,2)$, namely

$$\psi_s = \psi(1,2) + \psi(2,1) = \boxed{}\boxed{}\,, \quad \psi_a = \psi(1,2) - \psi(2,1) = \begin{array}{c}\boxed{}\\ \boxed{}\end{array}$$

with $\psi(2,1) = \widehat{P}_{12}\psi(1,2)$. Both, ψ_s as well as ψ_a, are eigenstates of the permutation operator $\widehat{P}_{12}$. Here each particle is associated with a box; two boxes in one row describe a symmetric state; two boxes in one column describe an antisymmetric state.

▶ We can define the symmetrizer $\widehat{S}_{12}$ and antisymmtrizer $\widehat{A}_{12}$ as

$$S_{12} = e + P_{12}\,, \quad A_{12} = e - P_{12}$$

Then

$$\begin{aligned} S_{12}\psi(1,2) =& (e + P_{12})\psi(1,2) = \psi(1,2) + \psi(2,1) = \psi_s \\ A_{12}\psi(1,2) =& (e - P_{12})\psi(1,2) = \psi(1,2) - \psi(2,1) = \psi_a \end{aligned}$$

▶ Anticipating our use of exchange of arbitrary particle label number $i \leftrightarrow j$ we define the general symmetrizers and antisymmetrizers

$$S_{ij} = e + P_{ij} \quad A_{ij} = e - P_{ij}$$

To obtain the state with the required symmetry property from a Young Tableau we define the Young operator

$$\widehat{Y} = \left(\sum_{columns} A_\nu\right)\left(\sum_{rows} S_\lambda\right)$$

which then acts on $\psi(1,2,3,\ldots)$.

▶ Some examples of Young operators are:

★ Totally symmetric Young diagram

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array} \quad\to\quad \widehat{Y} = (e + (12))$$

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array} \quad\to\quad \widehat{Y} = (e + (12) + (13) + (23) + (123) + (132))$$

★ Totally antisymmetric Young diagram

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \quad\to\quad \widehat{Y} = (e - (12))$$

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array} \quad\to\quad \widehat{Y} = (e - (12) - (13) - (23) + (123) + (132))$$

★ Mix Young diagram

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad\to\quad \widehat{Y} = (e - (13))(e + (12))$$

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \quad\to\quad \widehat{Y} = (e - (12))(e + (13))$$

5.1.2 The connection between $\mathsf{SU}(2)$ and S_N

▶ To illustrate the basic techniques involved, we consider the spin state of a **two-electron system**:

★ We know from our previous study of $\mathsf{SU}(2)$ that we have three symmetric states corresponding to the three possible orientations of the spin-triplet and an antisymmetric state corresponding to the spin singlet.

★ The fundamental representation of $\mathsf{SU}(2)$ is spanned by the basis vectors

$$\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ , \quad \beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

These vectors represent the two states of a particle with spin 1/2. These basis vectors can be represented by the Young tableaux consisting of one box, i.e.

$$\begin{array}{|c|}\hline \\ \hline\end{array} \ : \quad \begin{array}{|c|}\hline \alpha \\ \hline\end{array} \quad \text{and} \quad \begin{array}{|c|}\hline \beta \\ \hline\end{array}$$

If we erase the contents of the box, the Young diagram $\begin{array}{|c|}\hline \\ \hline\end{array}$ symbolizes both members of the doublet.

★ Both irreducible representations of the permutation group S_2 are represented by the Young diagrams

$$\textbf{symmetric irrep: } \begin{array}{|c|c|}\hline & \\ \hline\end{array}, \qquad \textbf{antisymmetric irrep: } \begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}$$

★ By numbering the boxes we perceive that these diagrams also symbolize irreducible representations of $\mathsf{SU}(2)$.

★ Let us start with the symmetric state. The boxes can have either α or β inside. Thus we get

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array} : \quad \begin{cases} \begin{array}{|c|c|}\hline \alpha & \alpha \\ \hline\end{array} = \alpha_1\alpha_1 = \alpha\alpha \\ \begin{array}{|c|c|}\hline \alpha & \beta \\ \hline\end{array} = \alpha_1\beta_2 + \alpha_2\beta_1 = \alpha\beta + \beta\alpha \\ \begin{array}{|c|c|}\hline \beta & \beta \\ \hline\end{array} = \beta_2\beta_2 = \beta\beta \end{cases}$$

The numerical labels represent the identification of particles, i.e. particle 1, 2 …. So α_2 means particle 2 in quantum state α. We get 3 possible states as we would expect from a triplet state! We do not need to consider $\begin{array}{|c|c|}\hline \beta & \alpha \\ \hline\end{array}$ because when we put boxes horizontally, symmetrization is understood. So we deduce an **important rule**: **Double counting is avoided if we require that the quantum numbers (label) do not decrease going from the left to the right, i.e. we are using $\alpha < \beta < \gamma < \ldots$.** From this argument we learn that the Young diagram $\begin{array}{|c|c|}\hline & \\ \hline\end{array}$ represents three different standard configurations, i.e. the corresponding multiplets have three dimensions.

★ As for the antisymmetric spin-singlet state we have

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \ : \quad \begin{array}{|c|}\hline \alpha \\ \hline \beta \\ \hline\end{array} = \alpha_1\beta_2 - \alpha_2\beta_1 = \alpha\beta - \beta\alpha$$

is the only possibility. Clearly $\begin{array}{|c|}\hline \alpha \\ \hline \alpha \\ \hline\end{array}$ and $\begin{array}{|c|}\hline \beta \\ \hline \beta \\ \hline\end{array}$ are impossible because of the requirement of antisymmetry. For a vertical tableaux we cannot have a symmetric state. Furthermore, $\begin{array}{|c|}\hline \beta \\ \hline \alpha \\ \hline\end{array}$ is discarded to avoid double counting. To eliminate the unwanted symmetry states, we therefore require the quantum number (label) to **increase as we go down**.

▶ We take the following to be the **general rule**

> In drawing Young tableaux, going from left to right the number cannot decrease; going down the number must increase.

We deduced this rule by considering the spin states of two electrons, but you can believe me or show your self that this rule is applicable for the construction of any tableaux.

▶ Let us now consider a **three-electron system**

★ We can construct a totally symmetric spin state using the previous rule

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}\, : \quad \begin{cases} \begin{array}{|c|c|c|}\hline \alpha & \alpha & \alpha \\ \hline\end{array} = \alpha\alpha\alpha \\ \begin{array}{|c|c|c|}\hline \alpha & \alpha & \beta \\ \hline\end{array} = \alpha\alpha\beta + \alpha\beta\alpha + \beta\alpha\alpha \\ \begin{array}{|c|c|c|}\hline \alpha & \beta & \beta \\ \hline\end{array} = \alpha\beta\beta + \beta\alpha\beta + \beta\beta\alpha \\ \begin{array}{|c|c|c|}\hline \beta & \beta & \beta \\ \hline\end{array} = \beta\beta\beta \end{cases}$$

This method gives four states altogether. This is just the multiplicity of the $j = 3/2$ state, which is obviously symmetric as seen from the $m = 3/2$ case, where all three spins are aligned up.

★ **What about the totally antisymmetric states?** We may try vertical tableaux as

$$\begin{array}{|c|}\hline \alpha \\ \hline \alpha \\ \hline \alpha \\ \hline\end{array} \quad \text{or} \quad \begin{array}{|c|}\hline \alpha \\ \hline \beta \\ \hline \beta \\ \hline\end{array}$$

But these are illegal, because the quantum numbers (labels) must increase as we go down. This is not surprising because total antisymmetry is impossible for spin states of three electrons; quite generally, a necessary (but not sufficient) condition for a total antisymmetry is that every state must be different. **In fact, in $SU(2)$ we cannot have three boxes in a vertical column.**

★ We now define a mixed symmetry tableaux that looks like $\begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1}\end{array}$. Such a tableaux can be visualized as either a single box attached to a symmetric tableaux

$$\begin{array}{|c|c|}\hline & \\ \hline \bullet \\ \cline{1-1}\end{array}$$

or a single box attached to an antisymmetric tableaux

$$\begin{array}{|c|c|}\hline & \bullet \\ \hline \\ \cline{1-1}\end{array}$$

★ In any case the dimensionality of $\begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$ is 2; that is, it represents a doublet ($j = 1/2$). So no matter how we consider it, $\begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$ must represent a doublet. But this is precisely what the rule gives. If the quantum number (label) cannot decrease in the horizontal direction and must increase in the vertical direction, the only possibilities are of arranging the quantum states are

$$\begin{array}{|c|c|}\hline \alpha & \alpha \\ \hline \beta & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad \text{and} \quad \begin{array}{|c|c|}\hline \alpha & \beta \\ \hline \beta & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

However, as we saw we could build two such doublets, i.e.

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad \text{and} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

Remember that the numerical labels identify the particles. We have three particles so we have labels going from 1 to 3. They can not be repeated and they always have to increase from left to right and up to down. (very similar to the quantum states labeling but with no repetition on the horizontal lines.)

★ We can now build these two doublets wavefunctions

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} = \left\{ \begin{array}{l} \begin{array}{|c|c|}\hline \alpha & \alpha \\ \hline \beta & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} = 2(\alpha\alpha\beta - \beta\alpha\alpha) \\[2ex] \begin{array}{|c|c|}\hline \alpha & \beta \\ \hline \beta & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} = 2(\alpha\beta\beta - \beta\beta\alpha) \end{array} \right.$$

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} = \left\{ \begin{array}{l} \begin{array}{|c|c|}\hline \alpha & \alpha \\ \hline \beta & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} = 2(\alpha\beta\alpha - \beta\alpha\alpha) \\[2ex] \begin{array}{|c|c|}\hline \alpha & \beta \\ \hline \beta & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} = 2(\alpha\beta\beta - \beta\alpha\beta) \end{array} \right.$$

The second doublet is antisymmetric in the first two particles by the first doublet is not symmetric in the first two. They are also not orthogonal. We can find such a basis. Let us take the first component of the doublets, i.e. (up to normalization)

$$\psi_a = \alpha\beta\alpha - \beta\alpha\alpha \quad \text{and} \quad \psi = \alpha\alpha\beta - \beta\alpha\alpha$$

we can define the orthogonal combination

$$\psi_s = \psi - \frac{\langle \psi | \, \psi_a \rangle}{\langle \psi_a | \, \psi_a \rangle} \psi_a = \psi - \frac{1}{2} \psi_a$$

We then get

$$\mathcal{M}_S = \begin{array}{|c|c|}\hline \; & \; \\ \hline \bullet & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array} \propto \begin{cases} -2\alpha\alpha\beta + \beta\alpha\alpha + \alpha\beta\alpha \\ 2\beta\beta\alpha - \alpha\beta\beta - \beta\alpha\beta \end{cases}$$

$$\mathcal{M}_A = \begin{array}{|c|c|}\hline \; & \bullet \\ \hline \; & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array} \propto \begin{cases} (\alpha\beta - \beta\alpha)\alpha \\ (\alpha\beta - \beta\alpha)\beta \end{cases}$$

where $\mathcal{M}_{S(A)}$ indicates mixed symmetry but remembering the pure symmetry (or antisymmetry) of the first two particles.

★ Because there are only two possibilities, $\begin{array}{|c|c|}\hline \; & \; \\ \hline \; & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}$ must correspond to a doublet. We can then consider the Clebsch-Gordan series (or angular momentum addition) as follows

$$\begin{cases} (1/2) \otimes (1/2) = (1) \oplus (0) \\ \begin{array}{|c|}\hline \; \\ \hline \end{array} \otimes \begin{array}{|c|}\hline \; \\ \hline \end{array} = \begin{array}{|c|c|}\hline \; & \; \\ \hline \end{array} \oplus \begin{array}{|c|}\hline \; \\ \hline \; \\ \hline \end{array} \end{cases} \qquad (\mathbf{2} \times \mathbf{2} = \mathbf{3} + \mathbf{1})$$

$$\begin{cases} (1) \otimes (1/2) = (3/2) \oplus (1/2) \\ \begin{array}{|c|c|}\hline \; & \; \\ \hline \end{array} \otimes \begin{array}{|c|}\hline \; \\ \hline \end{array} = \begin{array}{|c|c|c|}\hline \; & \; & \; \\ \hline \end{array} \oplus \begin{array}{|c|c|}\hline \; & \; \\ \hline \; & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array} \end{cases} \qquad (\mathbf{3} \times \mathbf{2} = \mathbf{4} + \mathbf{2})$$

$$\begin{cases} (0) \otimes (1/2) = (1/2) \\ \begin{array}{|c|}\hline \; \\ \hline \; \\ \hline \end{array} \otimes \begin{array}{|c|}\hline \; \\ \hline \end{array} = \begin{array}{|c|c|}\hline \; & \; \\ \hline \; & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array} \end{cases} \qquad \left(\text{Note: } \begin{array}{|c|}\hline \; \\ \hline \; \\ \hline \; \\ \hline \end{array} \text{ is impossible; } \mathbf{1} \times \mathbf{2} = \mathbf{2}\right)$$

▶ In order not to carry unnecessary boxes in the Young diagrams for $SU(2)$ since any two boxes in a column is a singlet we may just contract them and write

$$\begin{array}{|c|c|}\hline \; & \; \\ \hline \; & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array} \to \begin{array}{|c|}\hline \; \\ \hline \end{array}$$

5.1.3 Young diagrams for **SU**(3)

▶ We now extend our consideration to **three primitive objects**.

★ A box can assume three possibilities now

$$\begin{array}{|c|}\hline \; \\ \hline \end{array} : \quad \begin{array}{|c|}\hline \alpha \\ \hline \end{array}, \quad \begin{array}{|c|}\hline \beta \\ \hline \end{array}, \quad \begin{array}{|c|}\hline \gamma \\ \hline \end{array}.$$

The labels 1, 2 and 3 may stand for the magnetic quantum numbers of p-orbitals in atomic physics or charge states of the pion π^+, π^0, π^-, or the u, d and s quarks in the $SU(3)$ classification of elementary particles.

Let us start by assuming that the rule inferred using two primitive objects can be generalized and work out such concepts as the dimensionality; we check to see whether everything is reasonable.

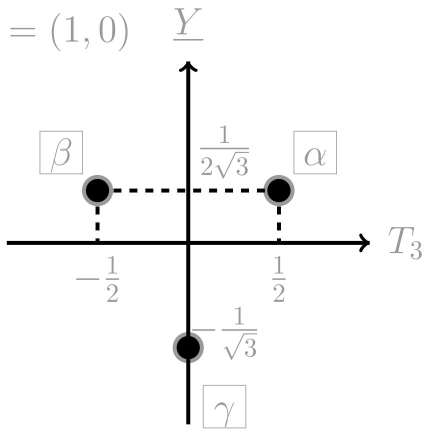
$$\boxed{1}\ :\ \boxed{\alpha}\qquad\boxed{\beta}\qquad\boxed{\gamma}\ :\ \text{dim } \mathbf{3}$$

$$\textbf{AntiSymm}\left\{\begin{array}{l}
\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\ :\ \begin{array}{|c|}\hline \alpha\\\hline \beta\\\hline\end{array}\quad \begin{array}{|c|}\hline \alpha\\\hline \gamma\\\hline\end{array}\quad \begin{array}{|c|}\hline \beta\\\hline \gamma\\\hline\end{array}\ :\ \text{dim } \mathbf{3}^* \quad (\text{to distinguish from } \mathbf{3}) \\[2ex]
\begin{array}{|c|}\hline \\\hline \\\hline \\\hline\end{array}\ :\ \begin{array}{|c|}\hline \alpha\\\hline \beta\\\hline \gamma\\\hline\end{array}\ :\ \text{dim } \mathbf{1} \quad (\text{Totally antisymmetrical})
\end{array}\right.$$

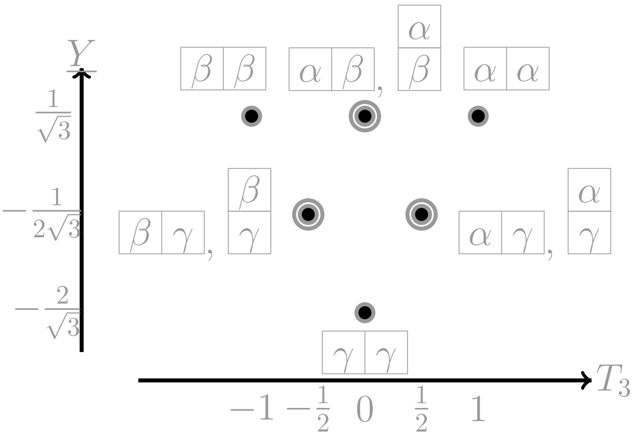
$$\textbf{Symm}\left\{\begin{array}{l}
\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}\ :\ \begin{array}{ccc}
\begin{array}{|c|c|}\hline \alpha&\alpha\\\hline\end{array} & \begin{array}{|c|c|}\hline \alpha&\beta\\\hline\end{array} & \begin{array}{|c|c|}\hline \alpha&\gamma\\\hline\end{array}\\
\begin{array}{|c|c|}\hline \beta&\beta\\\hline\end{array} & \begin{array}{|c|c|}\hline \beta&\gamma\\\hline\end{array} & \begin{array}{|c|c|}\hline \gamma&\gamma\\\hline\end{array}
\end{array}\ :\ \text{dim } \mathbf{6} \\[2ex]
\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\ :\ \begin{array}{ccc}
\begin{array}{|c|c|c|}\hline \alpha&\alpha&\alpha\\\hline\end{array} & \begin{array}{|c|c|c|}\hline \alpha&\alpha&\beta\\\hline\end{array} & \begin{array}{|c|c|c|}\hline \alpha&\alpha&\gamma\\\hline\end{array}\\
\begin{array}{|c|c|c|}\hline \alpha&\beta&\beta\\\hline\end{array} & \begin{array}{|c|c|c|}\hline \alpha&\beta&\gamma\\\hline\end{array} & \begin{array}{|c|c|c|}\hline \alpha&\gamma&\gamma\\\hline\end{array}\\
\begin{array}{|c|c|c|}\hline \beta&\beta&\beta\\\hline\end{array} & \begin{array}{|c|c|c|}\hline \beta&\beta&\gamma\\\hline\end{array} & \begin{array}{|c|c|c|}\hline \beta&\gamma&\gamma\\\hline\end{array}\\
\begin{array}{|c|c|c|}\hline \gamma&\gamma&\gamma\\\hline\end{array} & &
\end{array}\ :\ \text{dim } \mathbf{10}
\end{array}\right.$$

$$\textbf{Mixed}\left\{\begin{array}{l}
\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\\[1ex]
\text{or}\\[1ex]
\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}
\end{array}\right.\ :\ \begin{array}{cccc}
\begin{array}{|c|c|}\hline \alpha&\alpha\\\hline \beta\\\cline{1-1}\end{array} & \begin{array}{|c|c|}\hline \alpha&\beta\\\hline \beta\\\cline{1-1}\end{array} & \begin{array}{|c|c|}\hline \alpha&\gamma\\\hline \beta\\\cline{1-1}\end{array} & \begin{array}{|c|c|}\hline \alpha&\alpha\\\hline \gamma\\\cline{1-1}\end{array}\\[2ex]
\begin{array}{|c|c|}\hline \alpha&\beta\\\hline \gamma\\\cline{1-1}\end{array} & \begin{array}{|c|c|}\hline \alpha&\gamma\\\hline \gamma\\\cline{1-1}\end{array} & \begin{array}{|c|c|}\hline \beta&\beta\\\hline \gamma\\\cline{1-1}\end{array} & \begin{array}{|c|c|}\hline \beta&\gamma\\\hline \gamma\\\cline{1-1}\end{array}
\end{array}\ :\ \text{dim } \mathbf{8}$$

$\square = \mathbf{3} = (1,0)$



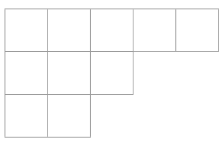
state	weight $(t_3, \underline{Y})$
$\boxed{\alpha}$	$\left(\frac{1}{2}, \frac{1}{2\sqrt{3}}\right)$
$\boxed{\beta}$	$\left(-\frac{1}{2}, \frac{1}{2\sqrt{3}}\right)$
$\boxed{\gamma}$	$\left(0, -\frac{1}{\sqrt{3}}\right)$



state	weight $(t_3, \underline{Y})$
$\boxed{\alpha\,\alpha}$	$\left(1, \frac{1}{\sqrt{3}}\right)$
$\boxed{\alpha\,\beta}$	$\left(0, \frac{1}{\sqrt{3}}\right)$
$\boxed{\beta\,\beta}$	$\left(-1, \frac{1}{\sqrt{3}}\right)$
$\boxed{\alpha\,\gamma}$	$\left(\frac{1}{2}, -\frac{1}{2\sqrt{3}}\right)$
$\boxed{\beta\,\gamma}$	$\left(-\frac{1}{2}, -\frac{1}{2\sqrt{3}}\right)$
$\boxed{\gamma\,\gamma}$	$\left(0, -\frac{2}{\sqrt{3}}\right)$

state	weight $(t_3, \underline{Y})$
$\boxed{\begin{matrix}\alpha\\ \beta\end{matrix}}$	$\left(0, \frac{1}{\sqrt{3}}\right)$
$\boxed{\begin{matrix}\alpha\\ \gamma\end{matrix}}$	$\left(\frac{1}{2}, -\frac{1}{2\sqrt{3}}\right)$
$\boxed{\begin{matrix}\beta\\ \gamma\end{matrix}}$	$\left(-\frac{1}{2}, -\frac{1}{2\sqrt{3}}\right)$

★ These tableaux correspond to the representations of $SU(3)$

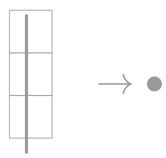


λ_1 boxes in first row

λ_2 boxes in second row

λ_3 boxes in third row

★ Once more, for the purpose of figuring out the dimensionality, it is legitimate to strike out



which is a **singlet**.

★ The dimensionality should be familiar at the stage and is given by

$$d(\lambda_1, \lambda_2, \lambda_3) = \frac{(p+1)(q+1)(p+q+2)}{2} \quad \text{with} \quad p = \lambda_1 - \lambda_2 \quad \text{and} \quad q = \lambda_2 - \lambda_3\,.$$

with (p, q) the same as the values we used to build the weight diagrams.

5.1.4 Dimensionality and Clebsch-Gordon series

Fundamental Theorem

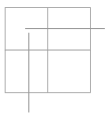
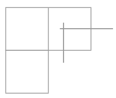
A tensor corresponding to a Young tableau of a given pattern forms the basis of an irreducible representation of $SU(n)$. Moreover if we enumerate all possible Young tableaux under the restriction that there should be no more than $n-1$ rows, the corresponding tensors form a complete set, in the sense that all finite-dimensional irreducible representations of the group are counted only once.

We next give two formulae of the **dimensionality of irreducible representations.** If the Young tableau is characterized by the length of its rows $[\lambda_1\lambda_2\cdots\lambda_{n-1}]$, define the length differences of adjacent rows as $f_1 = \lambda_1 - \lambda_2$, $f_2 = \lambda_2 - \lambda_3$, $\cdots$, $f_{n-1} = \lambda_{n-1}$ The dimension of an $SU(n)$ irreducible representation will then be the number of standard tableaux for a given pattern

$$
\begin{aligned}
d(f_1, f_2, \cdots, f_{n-1}) =& (1+f_1)(1+f_2)\cdots(1+f_{n-1}) \\
& \times \left(1+\frac{f_1+f_2}{2}\right)\left(1+\frac{f_2+f_3}{2}\right)\cdots\left(1+\frac{f_{n-2}+f_{n-1}}{2}\right) \\
& \times \left(1+\frac{f_1+f_2+f_3}{3}\right)\cdots\left(1+\frac{f_{n-3}+f_{n-2}+f_{n-1}}{3}\right) \\
& \cdots \\
& \times \left(1+\frac{f_1+f_2+\cdots+f_{n-1}}{n-1}\right)
\end{aligned}
$$

The formula above is rather cumbersome to use for large values of n; in such cases the second formulation is perhaps more useful. For this we need to introduce two definition:

- **hook length:** For any box in the tableau, draw two perpendicular lines, in shape of a **hook**, one going to the right and another going downward. The total number of boxes that this hook passes, including the original box itself, is the **hook length** (h_i) associated with the ith box. For example,

 $h_1 = 3$  $h_2 = 1$

- **distance to the first box:** this quantity, defined as D_i, is the number of steps going from the box in the upper left-handed corner of the tableau (the first box) to the ith box with each step towards the **right counted as $+1$** unit and each **downward step as -1** unit. For example, we have

$$\begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline -1 & 0 & \multicolumn{1}{c}{} \\ \cline{1-2} -2 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}$$

The dimension of the $SU(n)$-irreducible representation associated with the Young tableau is given by

$$d = \prod_i \frac{n + D_i}{h_i}\,.$$

The products are taken over all boxes in the tableau.

▶ Let us see some examples:

★ For $\mathsf{SU}(3)$ we have

$$\dim\left(\begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\right): \quad \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \Bigg/ \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} = \frac{3\times 4\times 2}{2\times 1\times 1} = 8$$

$$\dim\left(\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline\end{array}\right): \quad \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline\end{array} \Bigg/ \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline\end{array} = 1$$

★ In $\mathsf{SU}(6)$ we can have

$$\dim\left(\begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\right): \quad \begin{array}{|c|c|}\hline 6 & 7 \\ \hline 5 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \Bigg/ \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} = \frac{6\times 7\times 5}{2\times 1\times 1} = 70$$

$$\dim\left(\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline\end{array}\right): \quad \begin{array}{|c|}\hline 6 \\ \hline 5 \\ \hline 4 \\ \hline\end{array} \Bigg/ \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline\end{array} = \frac{6\times 5\times 4}{3\times 2\times 1} = 20$$

▶ Young diagrams may as we have seen can be used to reduce the product of $SU(N)$ irreps. Each of the two irreps being multiplied together is represented by its Young diagram. The squares of the smaller of the diagrams (less boxes) are filled with labels, the first row being labeled a, the second row b, the third row c, and so on. The labeled squares are then attached, one by one, to the larger diagram, forming new, partly labeled, Young diagrams. (As always, the lengths of the rows of any Young diagram cannot exceed the length of any higher row.)

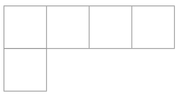
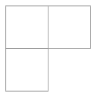
▶ The following restrictions apply at every stage:

- No two squares with the same label may occur in the same column;
- The total number of labels a, counting from right to left starting on the upper row and then moving down, cannot be less than the number of labels b, which itself cannot be less than the number of labels c, and so on, at each point in the reading process, i.e.

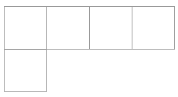
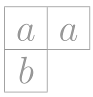
$$n_a \geq n_b \geq n_c \geq \cdots \quad \text{at any stage.}$$

- The same Young diagram may be produced more than once. If the labeling is the same in both diagrams, only one is retained. If the labeling is different, both are retained;
- For application to a specific $SU(n)$, diagrams with more than n rows are discarded and columns of n squares are removed from the diagrams.

Lets workout an example in $SU(3)$. Let us find the (Clebsch-Gordon) product of the two irreps

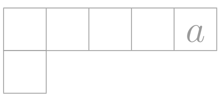
24 :  and **8** :  .

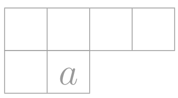
The first step is to identify the smallest diagram, i.e. **8**, and place it on the right side of the product with appropriate labeling

 $\otimes$ 

$=$

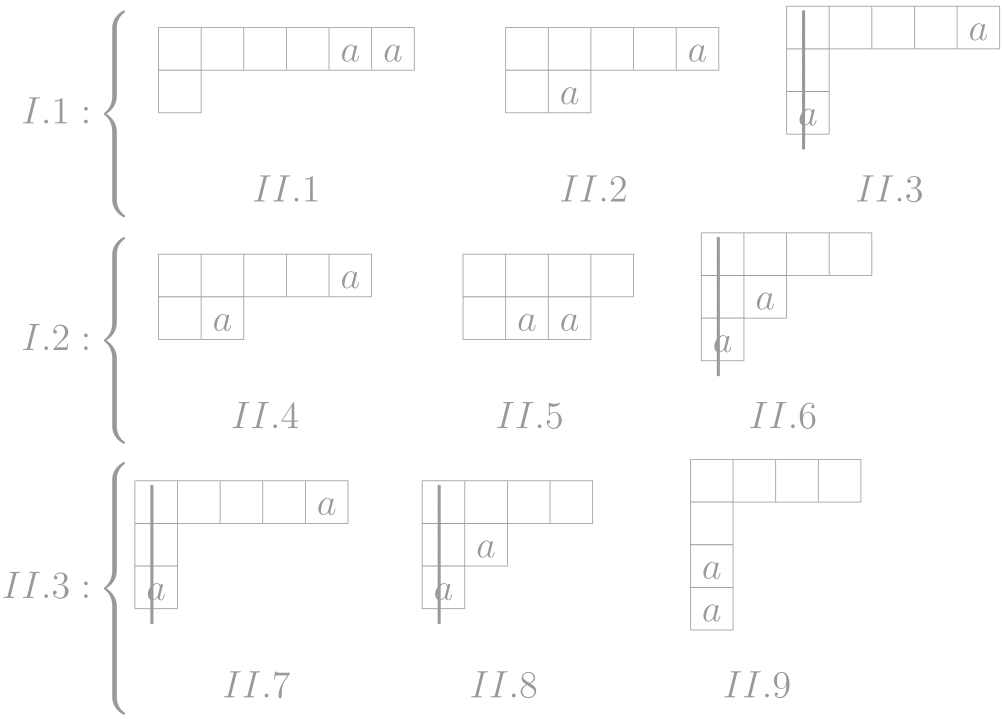
(I)





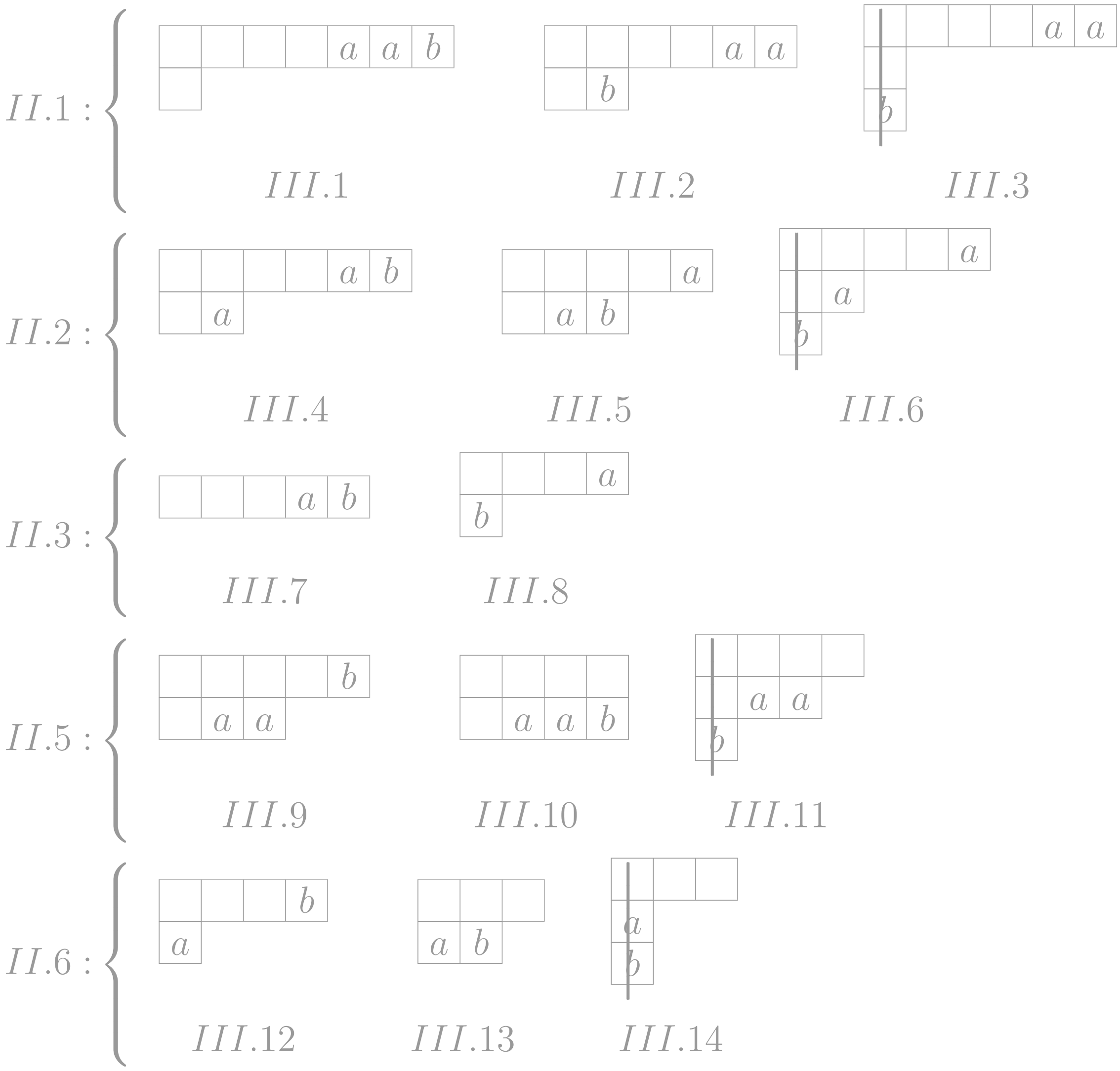
$I.1$ $I.2$ $I.3$

(II)



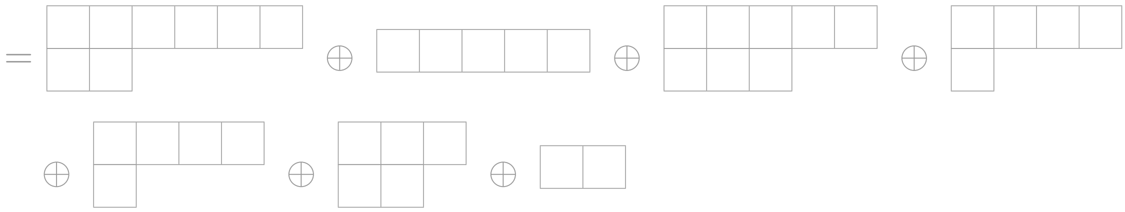
Cases $II.4$, $II.7$ and $II.8$ are the same Young tableaux as $II.2$, $II.3$ and $II.6$, respectively. The former (or latter) should be excluded. The case $II.9$ has more than 3 rows, it is not possible in $SU(3)$.

(III)



Cases $III.1$, $III.4$, $III.7$, $III.9$, $III.10$, $III.12$, $III.13$ read as $ba\cdots$, therefore are excluded.

Therefore we have



Therefore, we found

$$\mathbf{24} \otimes \mathbf{8} = \mathbf{60} \oplus \mathbf{21} \oplus \mathbf{42} \oplus \mathbf{24} \oplus \mathbf{24} \oplus \mathbf{15} \oplus \mathbf{6}$$

5.2 The $\mathsf{SU}(n-1)$ subgroups of $\mathsf{SU}(n)$

▶ The weight diagram of an $\mathsf{SU}(3)$ multiplet contains a number of $\mathsf{SU}(2)$ multiplets. Because of the symmetry of the diagrams there is no difference between submultiplets related to T-line or others related to $U-$ and $V-$line. We are now interested in finding such a decomposition.

▶ Let us take the octet in $\mathsf{SU}(3)$, i.e.

$$\begin{ytableau} \ & \ \\ \ \end{ytableau} = (1,1)$$

The eight corresponding Young tableaux are (NOTE: here we will use numbers to denote the states. There should be no confusion, since we are not interested in the wavefunctions. We are only interested in finding where the "last state", i.e. the state that is in $\mathsf{SU}(n)$ but not in $\mathsf{SU}(n-1)$ is. Instead of using a Greek letter, which can be difficult for large n, we use numbers.)

$$\begin{ytableau} 1 & 1 \\ 2 \end{ytableau} \quad \begin{ytableau} 1 & 2 \\ 2 \end{ytableau} \quad \begin{ytableau} 1 & 3 \\ 2 \end{ytableau} \quad \begin{ytableau} 1 & 1 \\ 3 \end{ytableau}$$

$$\begin{ytableau} 1 & 2 \\ 3 \end{ytableau} \quad \begin{ytableau} 1 & 3 \\ 3 \end{ytableau} \quad \begin{ytableau} 2 & 2 \\ 3 \end{ytableau} \quad \begin{ytableau} 2 & 3 \\ 3 \end{ytableau}$$

▶ In the case of $\mathsf{SU}(2)$, boxes containing the number 3 do not exist. Thus we divide the tableaux into groups according to the respective position of the boxes containing a 3:

★ First we find two tableaux that don't contain the number 3, i.e.

$$\begin{ytableau} 1 & 1 \\ 2 \end{ytableau} \quad \begin{ytableau} 1 & 2 \\ 2 \end{ytableau}$$

In $\mathsf{SU}(2)$ the column with two boxes can be omitted (it's a singlet), so we obtain a doublet

$$\begin{ytableau} 1 \end{ytableau} \quad \begin{ytableau} 2 \end{ytableau}$$

★ Now we look for a tableau with a single number on the right. There is only one of this kind

$$\begin{ytableau} 1 & 3 \\ 2 \end{ytableau}$$

Since the number 3 is meaningless in the case of $\mathsf{SU}(2)$, we can neglect this box and, thus obtain an $\mathsf{SU}(2)$ singlet

$$\begin{ytableau} 1 \\ 2 \end{ytableau} = 1$$

★ There are three tableaux with the number 3 at the bottom

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

and once again we can neglect the boxes containing the number 3, we obtain and $\mathsf{SU}(2)$ triplet

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline\end{array}$$

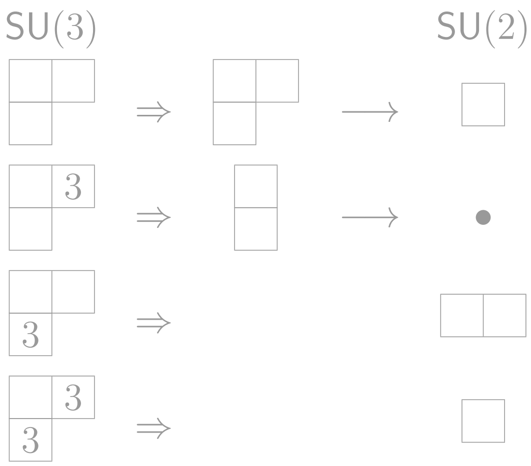
★ Finally we have two tableaux containing the number 3 twice

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

By erasing the three boxes containing the 3 we get a doublet

$$\begin{array}{|c|}\hline 1 \\ \hline\end{array} \qquad \begin{array}{|c|}\hline 2 \\ \hline\end{array}$$

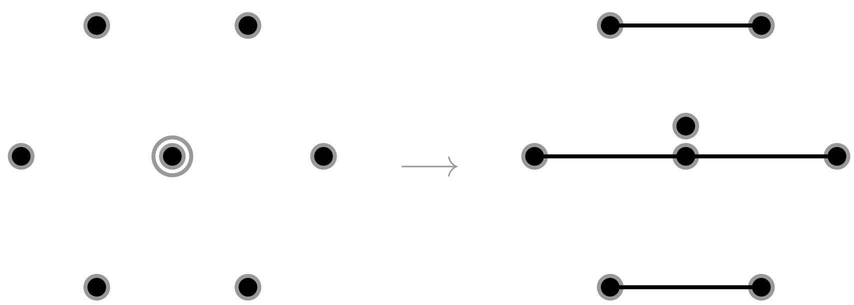
★ Summing all together, what we just did was



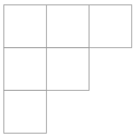
where in the first step we remove the boxes with 3 and in the second step contract two boxes in the same column.

We then have

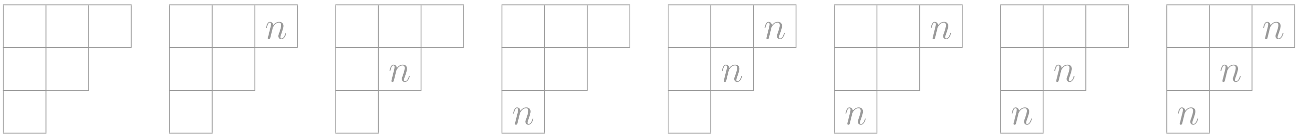
$$\begin{array}{ccc} \mathsf{SU}(3) & & \mathsf{SU}(2) \\ \mathbf{8} & \longrightarrow & \mathbf{1}+\mathbf{2}+\mathbf{2}+\mathbf{3} \end{array}$$



▶ What we just did can be generalized to the group $\mathsf{SU}(n)$. Consider all allowed positions of the boxes in the diagram containing the number n and remove these boxes. The $\mathsf{SU}(n-1)$ submultiplets are then given by the remaining boxes. For example, we consider the Young diagram

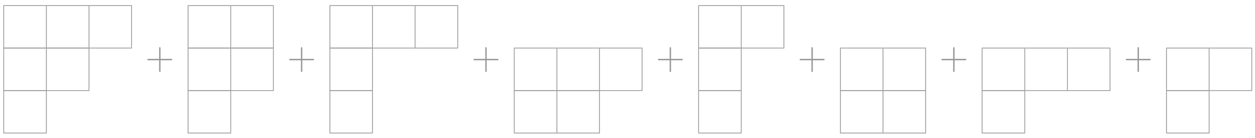


We have 8 ways of labeling the boxes with n



▶ The rule here is that n can occur only at the bottom of a column, because the numbers in the squares have to increase from top to bottom, and cannot decrease from right to left.

▶ We thus obtain the decomposition of the $\mathsf{SU}(n)$ multiplet into submultiplets of the group $\mathsf{SU}(n-1)$, i.e.



This decomposition can then be continued down to $\mathsf{SU}(n-2)$, and so on.

▶ We saw how to decompose the multiplets of $\mathsf{SU}(n)$ into multiplets of $\mathsf{SU}(n-1)$. There is however still and extra symmetry that can be added.

▶ Let us look back to the $\mathsf{SU}(3)$ case. If we choose to break down to $\mathsf{SU}(2)$ in the T-line direction we gave that, out of the general $\mathsf{SU}(3)$ generators we are piking only $T_{1,2,3}$, i.e.

$$T_i = \frac{1}{2}\begin{pmatrix} \sigma_i & \vec{0}^T \\ \vec{0} & 0 \end{pmatrix}$$

When acting of a vector (x, y, z) only the subspace xy "feels" the action of the group. There is however the generator Y, i.e.

$$Y = \frac{2}{\sqrt{3}}T_8 = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

which commutes with $\mathsf{SU}(2)_T$ generators. Therefore, it can generate a $U(1)$ group that may be preserved in this reduction process.

▶ We then can have $\mathsf{SU}(3) \to SU(2) \otimes U(1)$. The three components of the defining representation, the **3**, decompose into a doublet with hypercharge 1/3 and a singlet with hypercharge $-2/3$. In general, consider a Young tableau with n boxes and look at the components in which j indices transform like doublets and $n-j$ transform like singlets.

▶ The total hypercharge will be

$$Y_{young} = \frac{1}{3}j - \frac{2}{3}(n-j)$$

▶ We will denote the $n-j$ singlet components by a Young diagram of $n-j$ boxes in one row. This is the only valid Young tableau that we can build for these, because they all have the same index, and thus cannot appear in the same column.

▶ Lets see same examples:

★ The $\mathsf{SU}(3)$ **3**

$$\begin{array}{ccc} \ydiagram{1} & \to & (\ydiagram{1}, \bullet) \oplus (\bullet, \ydiagram{1}) \\ \mathbf{3} & \to & \mathbf{2}_{\frac{1}{3}} \oplus \mathbf{1}_{-\frac{2}{3}} \end{array}$$

★ The $\mathsf{SU}(3)$ $\bar{\mathbf{3}}$

$$\begin{array}{ccc} \ydiagram{1,1} & \to & (\ydiagram{1,1}, \bullet) \oplus (\ydiagram{1}, \ydiagram{1}) \\ \bar{\mathbf{3}} & \to & \mathbf{1}_{\frac{2}{3}} \oplus \mathbf{2}_{-\frac{1}{3}} \end{array}$$

★ The $\mathsf{SU}(3)$ **6**

$$\begin{array}{ccc} \ydiagram{2} & \to & (\ydiagram{2}, \bullet) \oplus (\ydiagram{1}, \ydiagram{1}) \oplus (\bullet, \ydiagram{2}) \\ \mathbf{6} & \to & \mathbf{3}_{\frac{2}{3}} \oplus \mathbf{2}_{-\frac{1}{3}} \oplus \mathbf{1}_{-\frac{4}{3}} \end{array}$$

★ The $\mathsf{SU}(3)$ **8**

$$\begin{array}{ccc} \ydiagram{2,1} & \to & (\ydiagram{2,1}, \bullet) \oplus (\ydiagram{1,1}, \ydiagram{1}) \oplus (\ydiagram{2}, \ydiagram{1}) \oplus (\ydiagram{1}, \ydiagram{2}) \\ \mathbf{8} & \to & \mathbf{2}_1 \oplus \mathbf{1}_0 \oplus \mathbf{3}_0 \oplus \mathbf{2}_{-1} \end{array}$$

▶ Now we can generalize the discussion of $SU(2) \otimes U(1) \subset SU(3)$. Consider the $SU(N) \times SU(M) \times U(1)$ subgroup of $SU(N+M)$ in which the $SU(N)$ acts on the first N indices and the $SU(M)$ acts on the last M indices. Both of these subgroups commute with a $U(1)$ which we can take to be M on the first N indices and N on the last M, i.e.

$$H \propto \begin{pmatrix} N\left\{\begin{matrix} M & & \\ & \cdots & \\ & & M \end{matrix}\right. & \\ & \left.\begin{matrix} -N & & \\ & \cdots & \\ & & -N \end{matrix}\right\}M \end{pmatrix}$$

▶ The fundamental as then the decomposition

$$\yng(1) = (\yng(1), \bullet)_M \oplus (\bullet, \yng(1))_{-N}$$

▶ For a general representation, the $U(1)$ charges are given by $nM - mN$, with n the number of boxes of the irrep in $SU(N)$ and m the number of boxes in the irrep of $SU(M)$. A trivial case is the fundamental. Other cases can be found by products of the fundamental irrep.

▶ For example

$$\yng(1) \otimes \yng(1) = \yng(2) \oplus \yng(1,1)$$

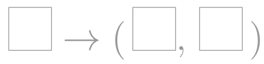
$$\begin{aligned} &\left[(\yng(1), \bullet) \oplus (\bullet, \yng(1))\right] \otimes \left[(\yng(1), \bullet) \oplus (\bullet, \yng(1))\right] \\ &= (\yng(2), \bullet) \oplus (\yng(1,1), \bullet) \oplus 2(\yng(1), \yng(1)) \oplus (\bullet, \yng(2)) \oplus (\bullet, \yng(1,1)) \end{aligned}$$

Thus

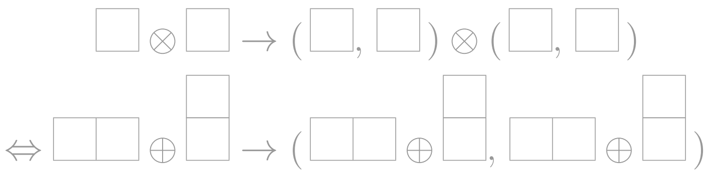
$$\begin{aligned} \mathsf{SU}(N+M) &\to \mathsf{SU}(N) \otimes \mathsf{SU}(M) \otimes U(1) \\ \yng(2) &\to (\yng(2), \bullet)_{2M} \oplus (\yng(1), \yng(1))_{M-N} \oplus (\bullet, \yng(2))_{-2N} \\ \yng(1,1) &\to (\yng(1,1), \bullet)_{2M} \oplus (\yng(1), \yng(1))_{M-N} \oplus (\bullet, \yng(1,1))_{-2N} \end{aligned}$$

▶ Notice that if we take $M = 1$, the $SU(M)$ group disappears, because there is no algebra $SU(1)$ - it has no generators. However, the construction described above still works. This is essentially what we used for the decomposition of $SU(3)$ representations under the $SU(2) \times U(1)$ subgroup.

▶ Another situation is $SU(N) \times SU(M) \in SU(NM)$. It arises only for $SU(k)$ where k is not a prime, and thus this embedding does not show up in $SU(2)$ or $SU(3)$. Under the $SU(N) \times SU(M)$ subalgebra generated by these matrices, the NM representation transforms like (N, M), i.e.

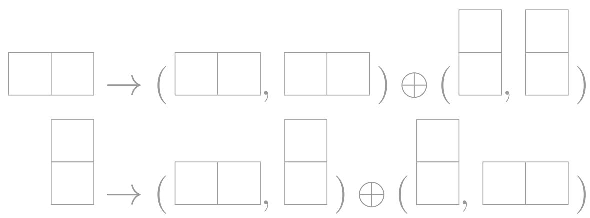


If we compute the product we get



Thus

$$\mathsf{SU}(NM) \to \mathsf{SU}(N) \otimes \mathsf{SU}(M)$$



The dimensions have to match for the original and the pieces.

5.3 Tensor and Young's diagrams

▶ Let ψ_a, ϕ_a, ... denote various spinors that transform as the basis of the fundamental representation:

$$\psi_a \to \psi'_a = S_a{}^b \psi_b \quad (a, b = 1, 2, 3)$$

under a transformation of $SU(N)$ with matrix elements S_{ab} . Covariant spinors spanning the space of the conjugate representation, which have components labeled by upper indices, must transform so as to make the inner product $\theta^a \psi_a$ invariant:

$$\theta^a \to \theta'^a = \theta^b \left(S^\dagger\right)_b{}^a \quad (a, b = 1, 2, 3)$$

▶ Spinors of the types ψ_a and θ^a are the simplest nontrivial examples of tensors. Generally, tensors are objects whose components, carrying both upper and lower indices, transform among themselves, with the upper indices transforming covariantly and the lower, contravariantly.

▶ If n stands for the number of upper indices, and m, the number of lower indices, the tensor will be denoted by $T(n,m)$ and its rank is defined by the ordered set of integers (n,m).

★ For instance, the mixed tensor of rank $(2,1)$ transforms as

$$T^a_{\ bc} \to T'^a_{\ bc} = S^a_{\ a'} T^{a'}_{\ b'c'} \left(S^\dagger\right)^{b'}_{\ \ b} \left(S^\dagger\right)^{c'}_{\ \ c} ,$$

while the tensor of rank $(1,2)$ transforms as

$$T^{ab}_{\ \ c} \to T'^{ab}_{\ \ c} = S^a_{\ a'} S^b_{\ b'} T^{a'b'}_{\ \ \ c'} \left(S^\dagger\right)^{c'}_{\ \ c} ,$$

The interest in these examples is that $T^a_{\ bc}$ obeys the same transformation rule as $\left(T^{bc}_{\ \ a}\right)^*$, a result which generalizes to tensors of arbitrary ranks, i.e. $T(n,m)$ transforms exactly as $T^*(m,n)$.

▶ There exist three **invariant tensors**, whose components are unchanged under the transformations of $\mathsf{SU}(3)$:

★ **Kronecker delta**

$$\delta^a_{\ b} = \begin{cases} 1\,, & \text{if } a = b\,, \\ 0, & \text{otherwise}\,. \end{cases}$$

★ **Levi-Cività**

$$\epsilon_{abc} = \begin{cases} 1\,, & \text{if } a,b,c \text{ is an even permutation of 1,2,3}\,, \\ -1\,, & \text{if } a,b,c \text{ is an odd permutation of 1,2,3}\,, \\ 0, & \text{otherwise}\,. \end{cases}$$

★ **Contravariant Levi-Cività** ϵ^{abc}, which is numerically equal to ϵ_{abc}, so that $\epsilon^{123} = +1$ and

$$\epsilon_{abm}\epsilon^{mcd} = \delta^c_{\ a}\delta^d_{\ b} - \delta^d_{\ a}\delta^c_{\ b}\,.$$

▶ As we have already see, in $\mathsf{SU}(3)$ the fundamental representation is not equivalent to its conjugate. Therefore, a covariant spinor θ_a is not linearly related to a contravariant spinor, and vice versa.

▶ It is rather related to an antisymmetric second-rank contravariant tensor

$$\theta^a = \epsilon^{abc}\psi_b\phi_c$$

That θ^a is indeed a first-rank contracovariant tensor follows from the invariance of ϵ^{abc}.

► Therefore a lower index cannot be made equivalent to an upper index (contrarily to SU(2)), and there must exit mixed tensors carrying indices of both types.

Lets see some examples:

★ The irreducible tensor $(1,1)$

$$T^a_{\ b} \to T^a_{\ b} - \frac{1}{3}\delta^a_{\ b} T^a_{\ a}$$

★ The irreducible tensor $(2,0)$

$$T_{ab} \to \underbrace{\frac{1}{2}\left(T_{ab} + T_{ba}\right)}_{S_{ab}} + \underbrace{\frac{1}{2}\left(T_{ab} - T_{ba}\right)}_{A_{ab}}$$

These two objects, S_{ab} and A_{ab}, transform as tensors and the transformations do not mix them. Therefore, the general tensor T^{ab} can be decomposed into two irreducible tensor, the 2-index symmetric and two-index antisymmetric one.

► Thus, irreducible tensor of $SU(3)$ are **traceless** and **totally symmetric** in the indices of the same type. Because of these restrictions, not all components of an irreducible tensor are linearly independent.

(n,m)	dim	Tableau	Tensor
$(0,0)$	$\mathbf{1}$	$\ydiagram{1,1,1}$	$1(\epsilon^{abc})$
$(1,0)$	$\mathbf{3}$	$\ydiagram{1}$	T_a
$(0,1)$	$\bar{\mathbf{3}}$	$\ydiagram{1,1}$	T^a
$(2,0)$	$\mathbf{6}$	$\ydiagram{2}$	T_{ab}
$(0,2)$	$\bar{\mathbf{6}}$	$\ydiagram{2,2}$	T^{ab}
$(1,1)$	$\mathbf{8} = \bar{\mathbf{8}}$	$\ydiagram{2,1}$	$T_a^b \left(\sum_{a=1}^3 T_a^a = 0\right)$
$(3,0)$	$\mathbf{10}$	$\ydiagram{3}$	T_{abc}
$(0,3)$	$\overline{\mathbf{10}}$	$\ydiagram{3,3}$	T^{abc}
$(2,1)$	$\mathbf{15}$	$\ydiagram{3,1}$	$T_{ab}^c \left(\sum_{a=0}^3 T_{ab}^a = 0\right)$
$(1,2)$	$\overline{\mathbf{15}}$	$\ydiagram{3,2}$	$T_a^{bc} \left(\sum_{a=0}^3 T_a^{ab} = 0\right)$
$(4,0)$	$\mathbf{15}'$	$\ydiagram{4}$	T_{abcd}
$(0,4)$	$\overline{\mathbf{15}}'$	$\ydiagram{4,4}$	T^{abcd}
$(3,1)$	$\mathbf{24}$	$\ydiagram{4,1}$	$T_{abc}^d \left(\sum_{a=1}^3 T_{abc}^a = 0\right)$
$(1,3)$	$\overline{\mathbf{24}}$	$\ydiagram{4,3}$	$T_a^{bcd} \left(\sum_{a=1}^3 T_a^{abc} = 0\right)$
$(2,2)$	$\mathbf{27} = \overline{\mathbf{27}}$	$\ydiagram{4,2}$	$T_{ab}^{cd} \left(\sum_{a=1}^3 T_{ab}^{ad} = 0\right)$

► Let us start with the product of a fundamental and anti-fundamental representation, i.e. $\bar{\mathbf{3}} \otimes \mathbf{3}$

$$\psi_a \sim \mathbf{3} \sim \ydiagram{1}, \quad \psi^a \sim \bar{\mathbf{3}} \sim \ydiagram{1,1}$$

$$\begin{array}{ccccccc} \psi^a\psi_b & & & = & \frac{1}{3}\delta^a{}_b\psi^a\psi_a & + & \left(\psi^a\psi_b - \frac{1}{3}\delta^a{}_b\psi^a\psi_a\right) \\ \bar{\mathbf{3}} & \otimes & \mathbf{3} & = & \mathbf{1} & \oplus & \mathbf{8} \\ \ydiagram{1} & & \ydiagram{1,1} & & \ydiagram{1,1,1} & & \ydiagram{2,1} \end{array}$$

We the have

$$\begin{aligned} S =& \frac{1}{\sqrt{3}}\psi^a\psi_a \\ M^a{}_b =& \psi^a\psi_b - \frac{1}{3}\delta^a{}_b\psi^a\psi_a \\ =& \begin{pmatrix} \frac{1}{3}\left(2\psi^1\psi_1 - \psi^2\psi_2 - \psi^3\psi_3\right) & \psi^1\psi_2 & \psi^1\psi_3 \\ \psi^2\psi_1 & \frac{1}{3}\left(-\psi^1\psi_1 + 2\psi^2\psi_2 - \psi^3\psi_3\right) & \psi^2\psi_3 \\ \psi^3\psi_1 & \psi^3\psi_2 & \frac{1}{3}\left(-\psi^1\psi_1 - \psi^2\psi_2 + 2\psi^3\psi_3\right) \end{pmatrix} \end{aligned}$$

► Let us look at the product of two fundamental representations

$$\begin{array}{ccccccc} \psi_a\psi_b & & & = & \frac{1}{2}\left(\psi_a\psi_b + \psi_b\psi_a\right) & + & \frac{1}{2}\left(\psi_a\psi_b - \psi_b\psi_a\right) \\ \mathbf{3} & \otimes & \mathbf{3} & = & \mathbf{6} & \oplus & \bar{\mathbf{3}} \\ \ydiagram{1} & & \ydiagram{1} & & \ydiagram{2} & & \ydiagram{1,1} \end{array}$$

Chapter 6

Is it Space? Is it Time? NO! It's Space-Time!

Lorentz and Poincaré Groups

6.1 Lorentz group

▶ The **Lorentz group** is the set of all transformations that preserve the inner product of Minkowski space

$$x^\mu x_\mu = x^\mu \eta_{\mu\nu} x^\nu = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 \,, \quad \text{with} \quad \eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

This is nothing more than the definition of

$$\mathsf{O}(1,3) = \left\{ \Lambda \in \mathsf{GL}(4,\mathbb{R}) | \quad \Lambda^T \eta \Lambda = \eta \right\} \tag{6.1}$$

with η being the **metric**. In other words, the Lorentz group preserves the metric.

★ From the definition above we have

$$\underbrace{\det(\Lambda^T \eta \Lambda)}_{\det(\Lambda)\det(\eta)\det(\Lambda)=-1} = \underbrace{\det(\eta)}_{-1} \quad \Rightarrow \quad \det(\Lambda) = \pm 1$$

★ For $\mu = \nu = 0$

$$\Lambda_0{}^\sigma \eta_{\sigma\rho} \Lambda^\rho{}_0 = \left(\Lambda^0{}_0\right)^2 - \sum_i \left(\Lambda^0{}_i\right)^2 = 1 = \eta_{00}$$

Thus

$$\left(\Lambda^0{}_0\right)^2 \geq 1 \quad \Rightarrow \quad \Lambda^0{}_0 \geq 1 \text{ or } \Lambda^0{}_0 \leq -1$$

With $^\uparrow$ denoting orthochronous (preserves the direction of time) components, and $\pm$ denoting the determinant sign (if the group component is "proper" or not), we can divide the Lorentz group into four components:

$$\left.\begin{array}{l}\det(\Lambda)=\pm 1\\ \text{sign}\left(\Lambda^0_0\right)=\pm 1\end{array}\right\}\quad\longrightarrow\quad\begin{cases}\det(\Lambda)=+1,\ \Lambda^0_0\geq 1: & \mathsf{SO}(1,3)^\uparrow\,(\mathsf{SO}(1,3)^\uparrow_+)\\ \det(\Lambda)=+1,\ \Lambda^0_0\leq -1: & \Lambda_P\Lambda_T\mathsf{SO}(1,3)^\uparrow\,(\mathsf{SO}(1,3)^\downarrow_+)\\ \det(\Lambda)=-1,\ \Lambda^0_0\geq 1: & \Lambda_P\mathsf{SO}(1,3)^\uparrow\,(\mathsf{SO}(1,3)^\uparrow_-)\\ \det(\Lambda)=-1,\ \Lambda^0_0\leq -1: & \Lambda_T\mathsf{SO}(1,3)^\uparrow\,(\mathsf{SO}(1,3)^\downarrow_-)\end{cases}$$

with $\mathsf{SO}(1,3)^\uparrow$ named the **proper orthochronous Lorentz group** or **restricted Lorentz group**. In the standard representation, the parity transformation and time reversal operations are denoted by

$$\text{Parity: }\Lambda_P=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\qquad\text{Time-reversal: }\Lambda_T=\begin{pmatrix}-1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

▶ The 4 components are disconnected in the sense that it is not possible to get a Lorentz transformation of another component just by using transformations of one component.

▶ We can restrict our search for representations of the Lorentz group, to representations of $\mathsf{SO}(1,3)^\uparrow$ and combine this with representations for $\Lambda_{P,T}$.

6.1.1 Group and Algebra structure

▶ Let us use the defining relation of the Lorentz group to construct an explicit matrix representation.

▶ First let's think for a moment about what we are trying to do. The Lorentz group, when acting on 4-vectors (Minkowski space $\mathbb{R}^{(1,3)}$), is given by real 4×4 matrices.

▶ A generic real 4×4 matrix has 16 parameters. The defining condition of the Lorentz group, which is in fact 10 conditions, restricts this to 6 parameters. This is the numbers of linearly independent generators. ($N(N-1)/2$ for generic orthogonal groups.)

▶ Let us then find these generators:

★ First note that the rotation matrices of 3D Euclidean space leave time unchanged, we have

$$-R^T\mathbb{I}_3R = -R^TR = -\mathbb{I}_3$$

which is the defining condition for $\mathsf{O}(3)$. Together with $\det(\Lambda) = +1$ we have the $\mathsf{SO}(3)$ group. Such a Lorentz transformation is given by

$$\Lambda_{rot} = \begin{pmatrix} 1 & \vec{0} \\ \vec{0}^T & R_{3\times3} \end{pmatrix}$$

The corresponding infinitesimal generators (Lie algebra) are

$$A_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, A_y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, A_z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Later we will use the the hermitian combinations, i.e. $L_i = iA_i$.

★ Assuming that we can generate group elements with $e^{\alpha_a K^a}$ and expanding $\Lambda\eta\Lambda^T = \eta$, we have up to first order in α

$$\begin{aligned} (1+\alpha_a K^a)\eta(1+\alpha_b K^b)^T &= \eta \Rightarrow && (6.2) \\ K^a\eta &= -\eta(K^a)^T \qquad \text{for first order in } \alpha_a\ . && (6.3) \end{aligned}$$

A transformation generated by these generators is called a boost, i.e. a change into a coordinate system that moves with a different constant velocity.

★ Assume we boost in the x-axis. The generator takes the form

$$K_x = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We just need to solve the system for the 2×2 non-zero block, i.e.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} \rightarrow \begin{cases} a = -a \\ -c = -b \\ b = c \\ -d = d \end{cases} \rightarrow \begin{cases} a = d = 0 \\ b = c \end{cases}$$

Therefore, the infinitesimal generator of the boost along the x-axis (with the appropriate normalization) is

$$K_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and equivalently for the boosts along y- and z-axis

$$K_y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \quad K_z = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

★ Note that

$$\begin{aligned} \mathrm{Exp}\left[\phi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] &= \sum_{n=0}^{\infty} \frac{\phi^n}{n!} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^n = \sum_{n=0}^{\infty} \frac{\phi^{2n}}{(2n)!} \mathbb{I}_2 + \sum_{n=0}^{\infty} \frac{\phi^{2n+1}}{(2n+1)!} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \cosh\phi \mathbb{I}_2 + \sinh\phi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix} \end{aligned}$$

★ The group elements are

$$\Lambda_x = \begin{pmatrix} \cosh\phi & \sinh\phi & 0 & 0 \\ \sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \quad \Lambda_y = \begin{pmatrix} \cosh\phi & 0 & \sinh\phi & 0 \\ 0 & 1 & 0 & 0 \\ \sinh\phi & 0 & \cosh\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Lambda_z = \begin{pmatrix} \cosh\phi & 0 & 0 & \sinh\phi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\phi & 0 & 0 & \cosh\phi \end{pmatrix}$$

An arbitrary boost can be composed by multiplications of these 3 generators.

▶ To understand how the generators transform, we simply have to act with the parity operation Λ_P and the time reversal operator Λ_T on the matrices L_i, K_i:

$$\begin{array}{rrcl} \textsf{Parity:} & \Lambda_P L_i \Lambda_P^{-1} & = & L_i \\ & \Lambda_P K_i \Lambda_P^{-1} & = & -K_i \\ & & & \\ \textsf{Time-reversal:} & \Lambda_T L_i \Lambda_T^{-1} & = & L_i \\ & \Lambda_T K_i \Lambda_T^{-1} & = & -K_i \end{array} \tag{6.4}$$

This can be checked by counting minus signs appearing in the multiplication with $\Lambda_{P/T}$.

▶ Thus we have

$$L_i \quad \underbrace{\longrightarrow}_{P,T} \quad L_i \quad \text{and} \quad K_i \quad \underbrace{\longrightarrow}_{P,T} \quad -K_i$$

▶ The proper orthochronous Lorentz transformations are given by

$$\Lambda = \exp\left[\vec{\theta}.\vec{A} + \vec{\Phi}.\vec{K}\right] = \exp\left[-i\left(\vec{\theta}.\vec{L} + \vec{\phi}.\vec{K}'\right)\right] .$$

with $K_i' = iK_i$ and $L_i = iA_i$

▶ Using the explicit form of the generators for $\mathsf{SO}(1,3)$ we can derive the corresponding Lie algebra

$$[L_i, L_j] = i\epsilon_{ijk}L_k\,, \quad [L_i, K_j'] = i\epsilon_{ijk}K_k'\,, \quad [K_i', K_j'] = -i\epsilon_{ijk}L_k$$

▶ The two type of generators do not commute with each other. While the rotation generators close under commutation, the boost generators do not.

▶ We can define two sets of operators from the old ones that do close under commutation and commute with each other

$$N_i^{\pm} = \frac{L_i \pm iK_i'}{2}$$

which lead to the commutation relations

$$\underbrace{[N_i^+, N_j^+] = i\epsilon_{ijk}N_k^+}_{\mathfrak{su}(2)_+}\,, \quad \underbrace{[N_i^-, N_j^-] = i\epsilon_{ijk}N_k^-}_{\mathfrak{su}(2)_-}\,, \quad [N_i^+, N_i^-] = 0$$

▶ We have, therefore, discovered that the Lie algebra $\mathfrak{so}(1,3)_+^{\uparrow}$ consists of two copies of the Lie algebra $\mathfrak{su}(2)$

▶ This is great news, because we know how to construct all irreducible representations of the $\mathfrak{su}(2)$ Lie algebra. Note, however, that the group $\mathsf{SU}(2)_+ \otimes \mathsf{SU}(2)_-$ is compact but the Lorentz group is not. Even though the two groups share the same Lie algebra, one corresponds to the exponentiation of $\{iN^+,\, iN^-\}$, whereas the other to the exponentiation of $\{iL, iK'\}$, in both cases with real coefficients in the exponent (real algebras).

▶ Although irreducible representations of one group give irreducible representations of the other, some important properties of the representation, such as unitarity, are not preserved.

▶ We note that, for any unitary representation, the generators must be hermitian. While the set $\{N^+, N^-\}$ has only hermitian operators the K' in $\{L, K'\}$ is anti-hermitian. The non-unitary nature of finite dimensional representations of the Lorentz group is an important consideration for physical applications.

▶ In particular, these representations cannot correspond to physical states, since all symmetry operations must be realized as unitary operators on the space of physical states.

▶ By deriving the irreducible representations of the Lie algebra of the Lorentz group, we find the irreducible representations of the covering group of the Lorentz group, if we put the corresponding generators into the exponential function.

▶ Some of these will be representations of the Lorentz group, but we will find more than that. Each irreducible representation of the $\mathfrak{su}(2)$ algebra can be labeled by the scalar value j of the Casimir operator. Therefore, we know that we can label the irreducible representations of the covering group, i.e. $\mathsf{SL}(2,\mathbb{C})$ of the Lorentz group by two integer or half integer numbers: (j_1, j_2) with $(2j_1+1)(2j_2+1)$ degrees of freedom.

▶ The Lie group $\mathsf{SL}(2;\mathbb{C})$ is the set of all complex 2×2 matrices with determinant equal to unity. The Lie algebra $\mathfrak{sl}(2;\mathbb{C})$ consists of all traceless complex 2×2 matrices. There are six basis elements if $\mathfrak{sl}(2;\mathbb{C})$ is viewed as a real Lie algebra. We choose the form

$$
\begin{aligned}
a_1 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad a_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\
a_4 &= i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad a_5 = i \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad a_6 = i \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
\end{aligned}
$$

Clearly, $a_{k+3} = ia_k$ $(k=1,2,3)$, so $\mathfrak{sl}(2;\mathbb{C})$ is six-dimensional as a real Lie algebra, but only three-dimensional as a complex one (it is then $\widetilde{\mathfrak{su}(2)}$, i.e the complexified

$\mathfrak{su}(2)$ algebra, or $\widetilde{\mathfrak{sl}(2;\mathbb{R})}$), as the six a_k are not linearly independent over the complex numbers.

The real Lie algebra $\mathfrak{sl}(2;\mathbb{C})$, which is itself a simple Lie algebra, has a complexification that is not simple, but rather semi-simple: $\widetilde{\mathfrak{sl}(2;\mathbb{C})} \cong \widetilde{\mathfrak{su}(2)} \oplus \widetilde{\mathfrak{su}(2)}$.

6.2 Lorentz group representations

6.2.1 The $(0,0)$ representation

▶ The lowest order representation is as for SU(2) trivial, because the vector space is 1D for both copies of the $\mathfrak{su}(2)$ Lie algebra. Our generators must therefore be 1×1 matrices and the only 1×1 matrices fulfilling the commutation relations are the trivial 0

$$N_i^+ = N_i^- = 0 \quad \rightarrow \quad e^{-i\alpha_i N_i^+} = e^{-i\beta_i N_i^-} = 1\,.$$

▶ Therefore, the $(0,0)$ representation of the Lorentz group acts on objects that do not change under Lorentz transformations. This is called the Lorentz **scalar representation**.

6.2.2 The $\left(\frac{1}{2},0\right)$ representation

▶ In this representation we use the 2D representation for one copy of the $\mathfrak{su}(2)$ algebra N_i^+, i.e. $N_i^+ = \sigma_i/2$, and the 1D representation for the other N_i^-, i.e. $N_i^- = 0$. Therefore we get

$$N_i^- = \frac{L_i - iK_i'}{2} = 0 \quad \Rightarrow \quad L_i = iK_i'\,.$$

Therefore

$$N_i^+ = \frac{\sigma_i}{2} = \frac{L_i + iK_i'}{2} = iK_i'$$

Therefore

$$K_i' = -\frac{i}{2}\sigma_i \quad \text{and} \quad L_i = \frac{1}{2}\sigma_i$$

The Lorentz group representations are then given by [1]

$$\textsf{Rotation: } R_\theta = e^{i\vec{\theta}.\vec{L}} = e^{i\vec{\theta}.\frac{\vec{\sigma}}{2}} \qquad\qquad \textsf{Boost: } \Lambda_\phi = e^{i\vec{\phi}.\vec{K}'} = e^{\vec{\phi}.\frac{\vec{\sigma}}{2}} \tag{6.5}$$

[1] Following our convention from previous sections, it would be more consistent to include a minus sign in the exponent, however, we omit this in our definition in (6.5).

For example, a rotation around the x-axis is given by

$$R_x(\theta) = e^{i\theta\frac{\sigma_1}{2}} = \begin{pmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \tag{6.6}$$

▶ One important thing to notice is that here we have complex 2×2 matrices, representing the Lorentz transformations. These transformation certainly do not act on the four vectors of Minkowski space, because these have 4 components.

▶ The two-component objects this representation acts on are called **left-chiral spinors**

$$\chi_L = \begin{pmatrix} (\chi_L)_1 \\ (\chi_L)_2 \end{pmatrix}$$

▶ Spinors in this context are two component objects. A possible definition for left-chiral spinors is that they are objects that transform under Lorentz transformations according to the $\left(\frac{1}{2}, 0\right)$ representation.

▶ Spinors, as we know, have properties that usual vectors do not have. For instance, the factor 1/2 in the exponent. This factor shows us that a spinor is after a rotation by 2π not the same but gets a minus sign.

6.2.3 The $\left(0, \frac{1}{2}\right)$ representation

▶ Let us now turn to the $\left(0, \frac{1}{2}\right)$ representation. This representation can be constructed analogous to the $\left(\frac{1}{2}, 0\right)$. This time we have

$$\begin{cases} N_i^+ = \dfrac{L_i + iK_i'}{2} = 0 \\ N_i^- = \dfrac{\sigma_i}{2} = \dfrac{L_i - iK_i'}{2} \end{cases} \longrightarrow \begin{cases} K_i' = \dfrac{i}{2}\sigma_i \\ L_i = \dfrac{1}{2}\sigma_i \end{cases}$$

▶ We then get

$$\text{Rotation: } R_\theta = e^{i\vec{\theta}.\vec{L}} = e^{i\vec{\theta}.\frac{\vec{\sigma}}{2}} \qquad\qquad \text{Boost: } \Lambda_\phi = e^{i\vec{\phi}.\vec{K}'} = e^{-\vec{\phi}.\frac{\vec{\sigma}}{2}}$$

▶ Therefore, rotations are the same as in the $\left(\frac{1}{2}, 0\right)$ representation, but boosts differ by a minus sign in the exponent. Therefore, both representations act on objects that are similar but not the same. We call these objects **right-chiral spinors**

$$\chi_R = \begin{pmatrix} (\chi_R)^1 \\ (\chi_R)^2 \end{pmatrix}$$

The generic name for left- and right-chiral spinors is **Weyl spinors**.

6.2.4 Van der Waerden notation

▶ There is a deep connection between the objects transforming according to the $\left(\frac{1}{2}, 0\right)$ representation (L spinors) and the objects transforming according to the $\left(0, \frac{1}{2}\right)$ representation (R spinors).

▶ In this section we will use the notation[2]

$$\text{left-chiral spinor: } \chi_L = \chi_a \qquad \text{right-chiral spinor: } \chi_R = \chi^{\dot{a}}\,. \tag{6.7}$$

We also define the complex conjugated versions of these fields

$$(\chi_L)^* = \chi_{\dot{a}} \qquad (\chi_R)^* = \chi^a \tag{6.8}$$

We will soon see why it makes sense to define the right-chiral spinor with a dotted index.

▶ We also introduce the spinor "metric", used in transforming spinors from one chirality to another

$$\epsilon^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{(Levi-Civita symbol)}$$

▶ Let us see how the **charge conjugated** field $\chi_L^C \equiv \epsilon\chi_L^*$ transforms under boosts

$$\begin{aligned}
\text{boost: } \chi_L^C \quad \rightarrow \quad \chi_L'^C =& \epsilon(\chi_L')^* \\
=& \epsilon(e^{\frac{1}{2}\vec{\phi}.\vec{\sigma}}\chi_L)^* \\
=& \epsilon(e^{\frac{1}{2}\vec{\phi}.\vec{\sigma}}\underbrace{(-\epsilon)(\epsilon)}_{=1}\chi_L)^* \\
=& \underbrace{\epsilon(e^{\frac{1}{2}\vec{\phi}.\vec{\sigma}^*}(-\epsilon)}_{\text{use: } \epsilon\sigma_i^*(-\epsilon)=-\sigma_i}\,\underbrace{\epsilon\chi_L^*}_{\chi_L^C}) \\
=& e^{-\frac{1}{2}\vec{\phi}.\vec{\sigma}}\chi_L^C
\end{aligned}$$

[2]Other notations frequently occur!

which is exactly the transformation behavior of a right-chiral spinor. The behavior under rotations is unchanged due to the additional i,

$$\text{rotation: } \chi_L^C \quad \rightarrow \quad \chi_L'^C = \epsilon(\chi_L')^* = \epsilon(e^{\frac{i}{2}\vec{\theta}.\vec{\sigma}}\chi_L)^* = e^{\frac{i}{2}\vec{\theta}.\vec{\sigma}}\epsilon(\chi_L)^*\,,$$

again as for right-chiral fields. We thus conclude that χ_L^C transforms as, i.e., is a right-chiral field.

▶ Since complex conjugation adds or removes a dot from an index, and since $\epsilon\chi_L^* = \chi_R$ we find that our "metric" raises or lowers an index

$$\epsilon\chi_L = \epsilon^{ac}\chi_c = \chi^a$$

This is analogous to the metric notation in special relativity.

▶ Let's see how to build a Lorentz invariant

★ From the transformation behavior of a left-chiral spinor

$$\chi_L = \chi_a \rightarrow \chi_a' = \left(e^{\frac{i}{2}\vec{\theta}.\vec{\sigma}+\frac{1}{2}\vec{\phi}.\vec{\sigma}}\right)_a^{\ b}\chi_b$$

we can derive how a spinor with a lower dotted index transforms

$$\begin{aligned}\chi_L^* = \chi_a^* = \chi_{\dot{a}} \rightarrow \chi_{\dot{a}}' = (\chi_a')^* &= \left[\left(e^{\frac{i}{2}\vec{\theta}.\vec{\sigma}+\frac{1}{2}\vec{\phi}.\vec{\sigma}}\right)_a^{\ b}\right]^*\chi_b^* \\ &= \left(e^{\frac{-i}{2}\vec{\theta}.\vec{\sigma}^*+\frac{1}{2}\vec{\phi}.\vec{\sigma}^*}\right)_{\dot{a}}^{\ \dot{b}}\chi_{\dot{b}}\end{aligned}$$

★ From the transformation behavior of a right-chiral spinor

$$\chi_R = \chi^{\dot{a}} \rightarrow \chi'^{\dot{a}} = \left(e^{\frac{i}{2}\vec{\theta}.\vec{\sigma}-\frac{1}{2}\vec{\phi}.\vec{\sigma}}\right)^{\dot{a}}_{\ \ \dot{b}}\chi^{\dot{b}}$$

we can derive how a spinor with an upper undotted index transforms

$$\begin{aligned}\chi_R^* = (\chi^{\dot{a}})^* = \chi^a \rightarrow \chi'^a = (\chi'^{\dot{a}})^* &= \left[\left(e^{\frac{i}{2}\vec{\theta}.\vec{\sigma}-\frac{1}{2}\vec{\phi}.\vec{\sigma}}\right)^{\dot{a}}_{\ \ \dot{b}}\right]^*(\chi^{\dot{b}})^* \\ &= \left(e^{\frac{-i}{2}\vec{\theta}.\vec{\sigma}^*-\frac{1}{2}\vec{\phi}.\vec{\sigma}^*}\right)^a_{\ \ b}\chi^b\end{aligned}$$

★ To be able to write products of spinors that do not change under Lorentz transformations, we need one more ingredient, $\vec{a}.\vec{b} = \vec{a}^T\vec{b}$. In the same spirit we must not forget to transpose one of the spinors in the spinor product.

★ The term $\chi^a\chi_a$ is invariant under Lorentz transformations

$$\begin{aligned}\chi^a\chi_a \to \chi'^a\chi'_a =& \left[\left(e^{-\frac{i}{2}\vec{\theta}.\vec{\sigma}^*-\frac{1}{2}\vec{\phi}.\vec{\sigma}^*}\right)^a{}_b\chi^b\right]\left[\left(e^{\frac{i}{2}\vec{\theta}.\vec{\sigma}+\frac{1}{2}\vec{\phi}.\vec{\sigma}}\right)_a{}^c\chi_c\right]\\ =&\chi^b\underbrace{\left[\left(e^{-\frac{i}{2}\vec{\theta}.\vec{\sigma}^\dagger-\frac{1}{2}\vec{\phi}.\vec{\sigma}^\dagger}\right)_b{}^a\left(e^{\frac{i}{2}\vec{\theta}.\vec{\sigma}+\frac{1}{2}\vec{\phi}.\vec{\sigma}}\right)_a{}^c\right]}_{\delta_b^c}\chi_c\\ =&\chi^c\chi_c\end{aligned}$$

★ In the same way we can combine an upper, dotted index with a lower, dotted index. In contrast, a term of the form $\chi^{\dot{a}}\chi_a = (\chi^{\dot{a}})^T\chi_a = \chi_R^T\chi_L$ is not invariant under Lorentz transformations (Check this). Therefore a term combining a left-chiral with a right-chiral spinor is not Lorentz invariant. We conclude that we must always combine an upper index with a lower index of the same type

$$^a{}_a \quad \text{or} \quad ^{\dot{a}}{}_{\dot{a}}$$

Or formulated differently, we must combine the complex conjugate of a right-chiral spinor with a left-chiral spinor

$$\chi_R^\dagger\chi_L = (\chi_R^*)^T\chi_L = (\chi^a)^T\chi_a = \chi^a\chi_a$$

or

$$\chi_L^\dagger\chi_R = (\chi_L^*)^T\chi_R = (\chi_{\dot{a}})^T\chi^{\dot{a}} = \chi_{\dot{a}}\chi^{\dot{a}}$$

▶ In addition, we have now another justification for calling ϵ^{ab} the spinor metric, since

$$\chi_a\chi^a = \chi_a\epsilon^{ab}\chi_b \quad (\text{Minkowski metric: } x_\mu y^\nu = x_\mu\eta^{\mu\nu}y_\nu)$$

After setting up this notation we can write the spinor "metric" with lowered indices as

$$\epsilon_{ab} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

because we need $(-\epsilon)$ to get from χ_R to χ_L.

▶ We can now write the two transformation operators as one object Λ. For example, when it has dotted indices we know it multiplies with a right-chiral spinor and we know which transformation operator to choose

$$\begin{aligned}\chi_R \to& \chi'_R = \chi'^{\dot{a}} = \Lambda^{\dot{a}}{}_{\dot{b}}\chi^{\dot{b}} = \left(e^{i\vec{\theta}.\frac{\vec{\sigma}}{2}-\vec{\phi}.\frac{\vec{\sigma}}{2}}\right)^{\dot{a}}{}_{\dot{b}}\chi^{\dot{b}}\\ \chi_L \to& \chi'_L = \chi'_a = \Lambda_a{}^b\chi_b = \left(e^{i\vec{\theta}.\frac{\vec{\sigma}}{2}+\vec{\phi}.\frac{\vec{\sigma}}{2}}\right)_a{}^b\chi_b\end{aligned}$$

Therefore

$$\Lambda_{(\frac{1}{2},0)} = \left(e^{i\vec{\theta}.\frac{\vec{\sigma}}{2}+\vec{\phi}.\frac{\vec{\sigma}}{2}}\right) \equiv \Lambda_a{}^b \quad \text{and} \quad \Lambda_{(0,\frac{1}{2})} = \left(e^{i\vec{\theta}.\frac{\vec{\sigma}}{2}-\vec{\phi}.\frac{\vec{\sigma}}{2}}\right) \equiv \Lambda^{\dot{a}}{}_{\dot{b}}$$

6.2.5 The $\left(\frac{1}{2}, \frac{1}{2}\right)$ representation

▶ For this representation we use the $2D$ representation, for both copies of the $\mathfrak{su}(2)$ Lie algebra N_i^+ and N_i^-. Let's first have a look at what kind of objects our representation acts on.

▶ The copies will not interfere with each other, because N_i^+ and N_i^- commute. Therefore, our objects will transform separately under both copies.

▶ Let's name the objects we want to examine v. This object will have 2 indices $v_a^{\dot{b}}$, each transforming under a separate two-dimensional copy of $\mathfrak{su}(2)$.

▶ Since both indices take on two values (because each representation is $2D$), our object v will have four components. Therefore, the objects can be 2×2 matrices, but it's also possible to enforce a four component vector form.

▶ First we look at the complex matrix choice. A general 2×2 matrix has 4 complex entries and therefore 8 free parameters. As noted above we only need 4. we can write any complex matrix M as a sum of a hermitian ($H^\dagger = H$) and an anti-hermitian ($A^\dagger = -A$) matrix: $M = H + A$. In addition, we will show in a moment that our transformations in this representation always transform a hermitian 2×2 matrix into another hermitian 2×2 matrix and equivalently an anti-hermitian matrix into another anti-hermitian matrix. This means that hermitian and anti-hermitian matrices are invariant subsets. Therefore, working with a general matrix here, corresponds to having a reducible representation. We can therefore assume that our irreducible representation acts on hermitian 2×2 matrices.

Instead of examining $v_a^{\dot{b}}$, we will look at $v_{a\dot{b}}$, because then we can use the Pauli matrices as a basis for hermitian matrices, i.e.

$$v_{a\dot{b}} = v_\nu \sigma^\nu_{a\dot{b}} = v_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + v_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + v_2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + v_3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

where $v_a^{\dot{b}} = v_\mu \sigma^\mu_{a\dot{c}} \epsilon^{\dot{b}\dot{c}}$. We therefore write a general hermitian matrix as

$$v_{a\dot{b}} = \begin{pmatrix} v_0 + v_3 & v_1 - iv_2 \\ v_1 + iv_2 & v_0 - v_3 \end{pmatrix}$$

We now have a look at how $v_{a\dot{b}}$ transforms, thus we get

$$\begin{aligned} v \to v' = v'_{a\dot{b}} =& \Lambda_a{}^c \Lambda_{\dot{b}}{}^{\dot{d}} v_{c\dot{d}} \\ =& \left(e^{i\vec{\theta}.\frac{\vec{\sigma}}{2}+\vec{\phi}.\frac{\vec{\sigma}}{2}}\right)_a{}^c \left(e^{-i\vec{\theta}.\frac{\vec{\sigma}^*}{2}+\vec{\phi}.\frac{\vec{\sigma}^*}{2}}\right)_{\dot{b}}{}^{\dot{d}} v_{c\dot{d}} \\ =& \left(e^{i\vec{\theta}.\frac{\vec{\sigma}}{2}+\vec{\phi}.\frac{\vec{\sigma}}{2}}\right)_a{}^c v_{c\dot{d}} \left(e^{-i\vec{\theta}.\frac{\vec{\sigma}^\dagger}{2}+\vec{\phi}.\frac{\vec{\sigma}^\dagger}{2}}\right)^{\dot{d}}{}_{\dot{b}} \\ =& \left(e^{i\vec{\theta}.\frac{\vec{\sigma}}{2}+\vec{\phi}.\frac{\vec{\sigma}}{2}}\right)_a{}^c v_{c\dot{d}} \left(e^{-i\vec{\theta}.\frac{\vec{\sigma}}{2}+\vec{\phi}.\frac{\vec{\sigma}}{2}}\right)^{\dot{d}}{}_{\dot{b}} \quad \textsf{hermitian} \end{aligned}$$

Let's boost along the z-axis

$$\begin{aligned} v_{a\dot{b}} \to v'_{a\dot{b}} =& \left(e^{\phi\frac{\sigma_3}{2}}\right)_a{}^c v_{c\dot{d}} \left(e^{\phi\frac{\sigma_3}{2}}\right)^{\dot{d}}{}_{\dot{b}} \\ =& \begin{pmatrix} e^{\frac{\phi}{2}} & 0 \\ 0 & e^{-\frac{\phi}{2}} \end{pmatrix} \begin{pmatrix} v_0+v_3 & v_1-iv_2 \\ v_1+iv_2 & v_0-v_3 \end{pmatrix} \begin{pmatrix} e^{\frac{\phi}{2}} & 0 \\ 0 & e^{-\frac{\phi}{2}} \end{pmatrix} \\ =& \begin{pmatrix} e^{\phi}(v_0+v_3) & v_1-iv_2 \\ v_1+iv_2 & e^{-\phi}(v_0-v_3) \end{pmatrix} \end{aligned}$$

Therefore, using

$$v'_{a\dot{b}} = \begin{pmatrix} v'_0+v'_3 & v'_1-iv'_2 \\ v'_1+iv'_2 & v'_0-v'_3 \end{pmatrix} = \begin{pmatrix} e^{\phi}(v_0+v_3) & v_1-iv_2 \\ v_1+iv_2 & e^{-\phi}(v_0-v_3) \end{pmatrix}$$

we get

$$\begin{aligned} &\to \; v'_0+v'_3 = e^{\phi}(v_0+v_3) = (\cosh\phi+\sinh\phi)(v_0+v_3) \\ &\to \; v'_0-v'_3 = e^{-\phi}(v_0-v_3) = (\cosh\phi-\sinh\phi)(v_0-v_3) \end{aligned}$$

The addition an subtraction of both equations yields

$$\to \; v'_0 = \cosh\phi v_0 + \sinh\phi v_3$$

$$\to \; v'_3 = \sinh\phi v_0 + \cosh\phi v_3$$

which is exactly what we get using the 4-vector formalism

$$\begin{pmatrix} v'_0 \\ v'_1 \\ v'_2 \\ v'_3 \end{pmatrix} = \begin{pmatrix} \cosh\phi & 0 & 0 & \sinh\phi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\phi & 0 & 0 & \cosh\phi \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} \cosh\phi v_0 + \sinh\phi v_3 \\ v_1 \\ v_2 \\ \sinh\phi v_0 + \cosh\phi v_3 \end{pmatrix}$$

This is true for an arbitrary Lorentz transformation.

▶ What we have shown here is that the $\left(\frac{1}{2},\frac{1}{2}\right)$ representation is the vector representation. We can simply transform our transformation laws by using the enforced vector form, because multiplying a matrix with a vector is simpler than the multiplication of 3 matrices.

▶ Nevertheless, we have seen how the familiar 4-vector is related to the more fundamental spinors. A 4-vector is a rank-2 spinor, which means a spinor with 2 indices that transforms according to the $\left(\frac{1}{2},\frac{1}{2}\right)$ representation of the Lorentz group.

▶ On can then write

$$D^{(\frac{1}{2},\frac{1}{2})} = D^{(\frac{1}{2},0)} \otimes D^{(0,\frac{1}{2})}$$

In general, one can obtain higher irreps by decomposition of direct products; it can be shown that the following relation holds

$$D^{(j_1,j_2)} \otimes D^{(j_1',j_2')} = D^{(j_1+j_1',j_2+j_2')} \oplus D^{(j_1+j_1'-1,j_2+j_2')} \oplus \ldots \oplus D^{(|j_1-j_1'|,|j_2-j_2'|)}$$

which is the analogue of the one found for $\mathsf{SU}(2)$.

▶ We see that an irrep of $\mathsf{SO}(1,3)^{\uparrow}$ contains, in general, several irreps of $\mathsf{SU}(2)$. We know that each element of the basis of D^j describes one of the $(2j+1)$ states of a particle with spin j. If we want to keep this correspondence between irreps and states with definite spin also in the case of $\mathsf{SO}(1,3)^{\uparrow}$, we have to introduce supplementary conditions which reduce the number of independent basis elements.

▶ If we want the basis of $D^{(j,j')}$ to describe a unique value of spin (we choose for it the highest value, since the lower ones can be described by lower irreps), we have to keep only $2(j+j')+1$ elements out of $(2j+1)(2j'+1)$, so that these can be grouped together to form the basis of $D^{(j+j')}$ in $SU(2)$.

▶ If one restricts oneself to the subgroup of rotations, the representations are no longer irreducible, and they can be decomposed in terms of the irreps of $\mathsf{SU}(2)$ as follows:

$$D^{(j,j')} \to D^{(j)} \otimes D^{(j')} = D^{(j+j')} \oplus \ldots \oplus D^{(|j-j'|)}$$

The number of conditions is then given by $4jj'$. For instance, in the simple case

$$D^{(\frac{1}{2},\frac{1}{2})} \to D^{(1)} \oplus D^{(0)}$$

we see that a spin 1 particle is described by a four-dimensional basis; so that one needs a supplementary condition (called in this case Lorentz condition) to leave only 3 independent elements which describes the 3 different spin 1 states.

6.2.6 Infinite-Dimensional Representations

At this point we already have every finite-dimensional irreducible representation we need for most problems. Nevertheless, there is another representation, the infinite-dimensional representation, that is especially interesting, because we need it to transform physical fields.

Finite-dimensional representations acted on constant one-, two- or four-dimensional objects so far. In most physical problems the objects we are dealing with are dynamically changing in space and time, so we need to understand how such objects transform. So far we have dealt with transformations of the form

$$\Phi_a \quad \to \quad \Phi'_a = M_{ab}(\Lambda)\Phi_b$$

where $M_{ab}(\Lambda)$ denotes the matrix of the particular finite-dimensional representation of the Lorentz transformation. The result of the multiplication with this matrix is simply that the components of the object in question get mixed and are multiplied with constant factors.

If our object Φ changes in space and time, it is a function of coordinates $\Phi \equiv \Phi(x)$ and these coordinades are affected by the Lorentz transformations, too. In general we have

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu$$

where $\Lambda^\mu{}_\nu$ denotes the vector representation ($\left(\frac{1}{2}, \frac{1}{2}\right)$ representation) of the Lorentz transformation in question. We have in this case

$$\Phi_a(x) \to M_{ab}(\Lambda)\Phi_a(\Lambda^{-1}x)$$

Our transformation will therefore consist of two parts. One part, represented by a finite-dimensional representation, acting of Φ_a and a second part acting on the coordinates. This second part will act on an infinite-dimensional vector space and we therefore need an infinite-dimensional representation. The infinite-dimensional representation of the Lorentz group is given by the differential operators

$$M^{\text{inf}}_{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu)$$

which satisfies the Lorentz algebra and transforms the coordinates as desired. The transformation of the coordinates is now given by

$$\Phi(\Lambda^{-1}x) = e^{-i\frac{\omega^{\mu\nu}}{2}M^{\text{inf}}_{\mu\nu}}\Phi(x)$$

The complete transformation is then a combination of a transformation generated by the finite-dimensional representation $M^{\text{fin}}_{\mu\nu}$ and a transformation generated by the infinitesimal representation $M^{\text{inf}}_{\mu\nu}$ of the generators

$$\begin{aligned}
\Phi_a(x) &\to \left(e^{-i\frac{\omega^{\mu\nu}}{2}M^{\text{fin}}_{\mu\nu}}\right)_a^{\ b} e^{-i\frac{\omega^{\mu\nu}}{2}M^{\text{inf}}_{\mu\nu}}\Phi_b(x) \\
&\to \left(e^{-i\frac{\omega^{\mu\nu}}{2}M_{\mu\nu}}\right)_a^{\ b} \Phi_b(x)
\end{aligned}$$

with $M_{\mu\nu} = M^{\text{fin}}_{\mu\nu} + M^{\text{inf}}_{\mu\nu}$. This is called field representation.

6.3 Poincaré group

▶ We can now talk about a different kind of transformation: translation, which means transformations to another location in spacetime. Translations do not result in mixing of components and therefore, we need no finite-dimensional representation, but it's quite easy to find the infinite-dimensional representation for translations. These are not part of the Lorentz group, but the laws of nature should be location independent. The Lorentz group (rotations and boosts) plus translations is called Poincare group.

Poincare group = Lorentz group + translations

▶ We know that the Lorentz group consists of four disjoint components, corresponding to the possible choices of the signs of $det\Lambda$ and Λ^0_0. Similarly, the Poicaré group consists of four disjoint components, each of which contains the corresponding component of the Lorentz group.

▶ We shall limit ourselves to the transformations of the proper orthochronous inhomogeneous Lorentz group, i.e translations and $\mathsf{SO}(1,3)^{\uparrow}$.

▶ For simplicity we restrict ourselves to one dimension. In this case an infinitesimal translation of a function, along the x-axis is given by

$$\Phi(x) \to \Phi(x+\epsilon) = \Phi(x) + \partial_x \Phi(x)\epsilon$$

which is, of course, again the first term of the Taylor series expansion. It is conventional to add an extra $-i$ and define

$$P_i \equiv -i\partial_i \,.$$

With this definition an arbitrary, finite translation is

$$\Phi(x) \to \Phi(x+a) = e^{-ia^i P_i}\Phi(x) = e^{a^\mu \partial_i}\Phi(x)$$

where a^i denotes the amount we want to translate in each direction. If we want to transform to another point in time we use $P_0 = i\partial_0$.

▶ The commutation relations of the infinitesimal generators are easily obtained giving a specific representation of the Lie algebra of $\mathcal{P}^{\uparrow}$. For this purpose it is convenient to write the group element as a 5×5 matrix

$$\begin{pmatrix} \Lambda & \vec{a}^T \\ 0 & 1 \end{pmatrix}$$

acting on the vector

$$\begin{pmatrix} \vec{x}'^T \\ 1 \end{pmatrix} = \begin{pmatrix} \Lambda & \vec{a}^T \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \vec{x}^T \\ 1 \end{pmatrix} = \begin{pmatrix} \Lambda \vec{x}^T + \vec{a}^T \\ 1 \end{pmatrix}$$

the infinitesimal generators for the translations are then given by

$$P_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and similar for P_3 and P_4. In the same representation, the generators L_i and K_i' are replaced by 5×5 matrices, by adding a fifth row and a fifth column of zeros.

► The generators of the Poicare group are the generators of the Lorentz group L_i, K_i plus the generators of translations in Minkowski space P_μ. The algebra reads

$$\begin{aligned} [L_i, L_j] =& i\epsilon_{ijk} L_k \,, \quad [L_i, K_j'] = i\epsilon_{ijk} K_k' \,, \quad [K_i', K_j'] = i\epsilon_{ijk} L_k \\ [L_i, P_j] =& i\epsilon_{ijk} P_k \,, \quad [J_i, P_0] = 0 \,, \quad [K_i', P_j] = -i\delta_{ij} P_0 \,, \quad [K_i', P_0] = -iP_i \end{aligned}$$

Because this looks like a huge mess it is conventional to write this in terms of $M_{\mu\nu}$, which was defined by

$$L_i = \epsilon_{ijk} M_{jk} \,, \quad K_i = M_{0i}$$

With $M_{\mu\nu}$ the Poincare algebra reads

$$\begin{aligned} [P_\mu, P_\nu] =& 0 \,, \\ [M_{\mu\nu}, P_\rho] =& -i(\eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu) \\ [M_{\mu\nu}, M_{\rho\sigma}] =& -i(\eta_{\mu\rho} M_{\nu\sigma} - \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\nu\sigma} M_{\mu\rho}) \end{aligned}$$

► For this quite complicate group it is very useful to label the representations by using the fixed scalar values of the Casimir operators. The Poincare group has two Casimir operators

$$\text{Casimir Operators:} \begin{cases} \text{momentum squared:} & P_\mu P^\mu \equiv m^2 \quad \text{(mass of particle)} \\ \text{Pauli-Lubanski 4-vector:} & W_\mu W^\mu \text{ with } W^\mu = \epsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma} \end{cases}$$

▶ We have exhibited a 5-dimensional representation of the proper orthochronous Poincaré group, which is not unitary, since both the generators P_μ and K_i' are expressed by non-hermitian matrices.

▶ This is actually a more general statement, since the group is not compact, no finite-dimensional unitary irreps exist.

▶ We recall from quantum mechanics the well known fact that the infinitesimal generators of translations P_μ can be identified with the energy-momentum operators. Moreover, the infinitesimal generators $M_{\mu\nu}$ can be identified with the components of the angular momentum tensor.

▶ For physical applications, we are interested in those irreps in which the operators P_μ and $M_{\mu\nu}$ are hermitian, since they correspond to dynamical variables, i.e. in the unitary, and hence infinite-dimensional, irreps of the Poincaré group.

▶ Representations in the Poincare group will then be labeled by two scalar values:

$$m \text{ (can take arbitrary values)} \quad \text{and} \quad j = j_1 + j_2 \text{ (half-integer or integer values)}$$

- **Spin 0** is described by an object Φ, called scalar, that transforms according to the $(0, 0)$, called spion 0 representation or scalar representation. (Ex: Higgs particle)

- **Spin 1/2** is described by an object Ψ, called spinor, that transforms according to the $\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)$ representation called spin $\frac{1}{2}$ representation or spinor representation. (Ex: electrons and quarks)

- **Spin 1** is described by an object A, called vector, that transforms according to the $\left(\frac{1}{2}, \frac{1}{2}\right)$, called spin 1 representation or vector representation. (Ex: Photons, gluons, $W^\pm$ and Z^0.)

The irreducible representations of the Poincare group are the mathematical tools we need to describe all elementary particles.

Much more could have been said about the Poincaré group . . .
For more details see H.F. Jones Chapter 10.