Symmetric Functions and the Symmetric Group 1

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He who can, does; he who cannot teaches.
George Bernard Shaw, *Man & Superman* (1903)
Those who can, do, those who can't, attend conferences.
Daily Telegraph 6th August, (1979)

Introduction

These are rough notes on symmetric functions and the symmetric group and are given purely as a guide. I intend to outline some of the basic properties of symmetric functions as relevant to application to problems in chemistry and physics. The partition of integers plays a key role and we shall first make remarks on partitions in order to establish notation and then go on to consider the standard symmetric functions, their definitions and their generators. This will lead to the important symmetric functions known as $S-$functions so named in honour of Schur. Important properties to be discussed will be their outer and inner multiplication and plethysm. At that stage we can start to look at specific applications.

Partitions

An *ordered* partition λ of *length* $p = \ell(\lambda)$, corresponds to an ordered set of p integers

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p) \tag{1}$$

such that

$$\lambda_1 \geq \lambda_2, \ldots \geq \lambda_p \geq 0 \tag{2}$$

Unless otherwise stated we shall mean by a partition an ordered partition. Normally we shall omit trailing zeros.

The *weight* ω_λ of a partition λ will be defined as the sum of its parts.

$$\omega_\lambda = |\ l| = \lambda_1 + \lambda_2, + \ldots + \lambda_p \tag{3}$$

If $|\ l| = n$ then λ is said to be a partition of n. We shall denot the set of partitions $\lambda \vdash n$ as $\mathcal{P}_n$ and the set of all partitions by $\mathcal{P}$. Thus

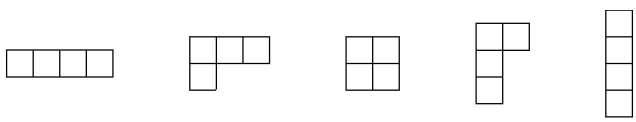
$$\mathcal{P}_4 \supseteq \{(4), (31), (2^2), (21^2), (1^4)\} \tag{4}$$

Note that the number of repetitions of a given part is often indicated by a superscript m_i where m_i is the number of parts of λ that are equal to i and will be referred to as the *multiplicity* of i in λ.

Note that in writing Eq.(4) we have given the partitions in *reverse lexicographic ordering* This ordering is such that for a pair of partitions (λ, μ) either $\lambda, \equiv \mu$ or the first non-vanishing difference $\lambda_i - \mu_i$ is *positive.*

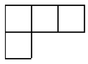
Frames of Partitions

We may associate with any partition λ a *frame* $\mathcal{F}^\lambda$ which consists of $\ell(\lambda)$, left-adjusted rows of boxes with the $i - th$ row containing λ_i boxes. Thus for $\mathcal{P}_4$ we have:-

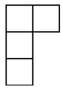


Conjugate Partitions

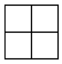
The *conjugate* of a partition λ is a partition λ' whose diagram is the transpose of the diagram of λ. If $\lambda' \equiv \lambda$ then the partition λ is said to be *self-conjugate*. Thus



and



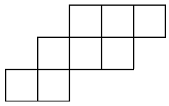
are conjugates while



is self-conjugate.

Skew Frames

Given two partitions λ and μ such that $\lambda \supset \mu$ implies that the frame F^{λ} contains the frame F^{μ}, i.e. that $\lambda_i \geq \mu_i$ for all $i \geq 1$. The difference $\rho = \lambda - \mu$ forms a *skew* frame $F^{\lambda/\mu}$. Thus, for example, the skew frame $F^{542/21}$ has the form



Note that a skew frame may consist of disconnected pieces.

Frobenius Notation for Partitions

There is an alternative notation for partitions due to Frobenius. The *diagonal* of nodes in a Ferrers-Sylvester diagram beginning at the top left-hand corner is called the *leading diagonal*. The number of nodes in the leading diagonal is called the *rank* of the partition. If r is the rank of a partition then let a_i be the number of nodes to the right of the leading diagonal in the $i-$th row and let b_i be the number of nodes below the leading diagonal in the $i-$th column. The partition is then denoted by Frobenius as

$$\begin{pmatrix} a_1, & a_2, & \ldots, & a_r \\ b_1, & b_2, & \ldots, & b_r \end{pmatrix} \tag{3.3}$$

We note that

$$a_1 > a_2 > \ldots > a_r$$
$$b_1 > b_2 > \ldots > b_r$$

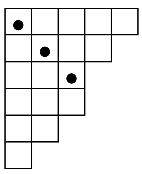
and

$$a_1 + a_2 + \ldots + a_r + b_1 + b_2 + \ldots + b_r + r = n$$

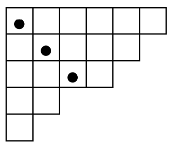
The partition conjugate to that of Eq.(3.3) is just

$$\begin{pmatrix} b_1, & b_2, & \ldots, & b_r \\ a_1, & a_2, & \ldots, & a_r \end{pmatrix} \tag{5}$$

As an example consider the partitions $(5\,4\,3^2\,2\,1)$ and $(6\,5\,4\,2\,1)$. Drawing their diagrams and marking their leading diagonal we have



and



from which we deduce the respective Frobenius designations

$$\begin{pmatrix} 4 & 2 & 0 \\ 5 & 3 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 5 & 3 & 1 \\ 4 & 2 & 0 \end{pmatrix}$$

Young Tableaux

A Young tableau is an assignment of n numbers to the n cells of a frame F^λ with $\lambda \vdash n$ according to some numbering sequence. A tableau is *standard* if the assignment of the numbers $1, 2, \ldots, n$ is such that the numbers are positively increasing from left to right in rows and down columns from top to bottom. Thus for the partitions of the integer 4 we have the standard Young tableaux

$$\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline\end{array}$$

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2\\\cline{1-1}\end{array}$$

$$\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array} \qquad \begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array}$$

$$\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1} 4\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1} 4\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1&4\\\hline 2\\\cline{1-1} 3\\\cline{1-1}\end{array}$$

$$\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline 4\\\hline\end{array}$$

We notice in the above examples that the number of standard tableaux for conjugate partitions is the same. Indeed the number of standard tableaux associated with a given frame F^λ is the *dimension* f_n^λ of an irreducible representation $\{\lambda\}$ of the symmetric group $\mathcal{S}_n$.

Hook lengths and dimensions for $\mathcal{S}_n$

The *hook length* of a given box in a frame F^λ is the length of the right-angled path in the frame with that box as the upper left vertex. For example, the hook length of the marked box in

$$\begin{array}{|c|c|c|c|c|}\hline \ & \ & \ & \ & \ \\\hline * & \cdot & \cdot & \cdot \\\cline{1-4} \cdot & & \\\cline{1-3} \cdot & & \\\cline{1-3} \cdot & \\\cline{1-2} \cdot \\\cline{1-1}\end{array}$$

is 8.

Theorem 1: *To find the dimension of the representation of $\mathcal{S}_n$ corresponding to the frame F^λ, divide $n\,!$ by the factorial of the hook length of each box in the first column of F^λ and multiply by the difference of each pair of such hook lengths.*

Thus for the partition $(5\,4\,3^2\,2\,1)$ we have the hook lengths

$$\begin{array}{|c|c|c|c|c|}\hline 10 & \ & \ & \ & \ \\\hline 8 & & & \\\cline{1-4} 6 & & \\\cline{1-3} 5 & & \\\cline{1-3} 3 & \\\cline{1-2} 1 \\\cline{1-1}\end{array}$$

and hence a dimension

$$\begin{aligned} f_{18}^{543^2 21} &= 18\,!\frac{2\times 4\times 5\times 7\times 9\times 2\times 3\times 5\times 7\times 1\times 3\times 5\times 2\times 4\times 2}{10\,!\times 8\,!\times 6\,!\times 5\,!\times 3\,!\times 1\,!} \\ &= 10720710 \end{aligned}$$

It is not suggested that you check the above result by explicit enumeration!

Hook-length Product $H^{\{\lambda\}}$

The irreps $\{\lambda\}$ of S_n are indexed by the ordered partitions $\lambda \vdash N$. It is useful to define a hook-length product

$$H^{\{\lambda\}} = \prod_{(i,j)\ni\lambda} h_{ij} \tag{6}$$

where i labels rows and j columns. Note that

$$H^{\{\lambda\}} = H^{\{\lambda\prime\}} \tag{7}$$

The Frame-Robinson-Thrall Formula

The $\mathcal{S}_n$ dimensional formula may be rewritten as

$$f_n^{\lambda} = \frac{n\,!}{H^{\{\lambda\}}} \tag{8}$$

which is the celebrated result of Frame, Robinson and Thrall.

Specialisation to Two-Row Irreps of $\mathcal{C}_n$

Consider a two-part partition (p, r). It is readily seen from the definition of $H^{\{\lambda\}}$ that

$$H^{\{p,r\}} = \frac{r!\,(p+1)!}{p-r+1} \tag{9}$$

Noting that $n = p + r$ we may specialise Eq. (8) to

$$f^{\{p,r\}} = \frac{p-r+1}{p+r+1}\binom{p+r+1}{r} \tag{10}$$

In quantum chemistry the Pauli exclusion principle restricts physically realisable irreps of $\mathcal{S}_n$ to the generic type $\{\frac{N}{2}+S, \frac{N}{2}-S\}$ where N and S are the total electron number and spin respectively. In that case Eq. (10) becomes

$$f^{(N,S)} = \frac{2S+1}{N+1}\binom{N+1}{\frac{N}{2}-S} \tag{11}$$

which is sometimes called the Heisenberg formula.

Staircase Partitions

A partition of the form $(p, p-1, p-2, \ldots, 2, 1)$ is termed a *staircase partition.* Such irreps have many interesting properties.

Exercises

- Show that the $p-th$ staircase partition is of weight

$$\frac{p(p+1)}{2} \tag{12}$$

- Show that the hooklength product H^p for the $p-th$ staircase partition is

$$H^p = \prod_{i=0}^{p-1}(2i+1)^{p-i} \tag{13}$$

- Show that the $p = 18$ staircase represention is of

$$353,630,151,029,664,166,403,885,519,184,771,102,250,561,450,895,264,176,910$$
$$,003,150,360,627,549,788,542,182,043,325,740,180,684,537,821,357,203,782,730$$
$$,400,746,242,708,749,607,205,510,228,035,502,080$$

- How long would it take a supercomputer to check this result by explicit computation?

Symmetric Functions and the Symmetric Group 2

B. G. Wybourne

With the odd number five strange natures laws
Plays many freaks nor once mistakes the cause
And in the cowslap peeps this very day
Five spots appear which time neer wears away
Nor once mistakes the counting - look within
Each peep and five nor more nor less is seen
And trailing bindweed with its pinky cup
Five lines of paler hue goes streaking up
And birds a many keep the rule alive
And lay five eggs nor more nor less then five
And flowers how many own that mystic power
With five leaves making up the flower
John Clare ˜ 1821

2.1 Permutations and the Symmetric Group

Permutations play an important role in the physics of identical particles. A permutation leads to a reordering of a sequence of objects. We can place n objects in the natural number ordering $1, 2, \ldots, n$. Any other ordering can be discussed in terms of this ordering and can be specified in a two line notation

$$\begin{matrix} 1 & 2 & \ldots & n \\ \pi(1) & \pi(2) & \ldots & \pi(n) \end{matrix} \tag{2.1}$$

For $n = 3$ we have the six permutations

$$\begin{matrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \\ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \end{matrix} \tag{2.2}$$

Permutations can be multiplied working from right to left. Thus

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

The six permutations in (2.2) satisfy the following properties:

1. There is an identity element $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$.
2. Every element has an inverse among the set of elements.
2. The product of any two elements yields elements of the set.
4. The elements satisfy the associativity condition $a(bc) = (ab)c$. These conditions establish that the permutations form a *group*. In general the $n!$ permutations form the elements of the *symmetric group* $\mathcal{S}_n$.

■ Exercise 2.1 Construct a multiplication table (The Cayley Table) for the six permutations given in (2.2) and verify that the set of six permutations form a group.

■ Exercise 2.2 Inspect your Cayley table and see what subsets of the elements satisfy the four group axioms and thus form a *subgroup* of $\mathcal{S}_6$.

■ 2.2 Cycle Structure of Permutations

It is useful to express permutations as a cycle structure. A cycle $(i, j, k, \ldots, l)$ is interpreted as $i \to j, j \to k$ and finally $l \to i$. Thus our six permutations have the cycle structures

$$(1)(2)(3), (1,2)(3), (1)(2,3), (1,3)(2), (1,3,2), (1,2,3) \tag{2.3}$$

The elements within a cycle can be cyclically permuted and the order of the cycles is irrelevant. Thus $(123)(45) \equiv (54)(312)$.

■ A *k-cycle* or *cycle of length k* contains k elements. It is useful to organise cycles into *types* or *classes*. We shall designate the *cycle type* of a permutation π by

$$(1^{m_1} 2^{m_2} \ldots, n^{m_n}) \tag{2.4}$$

where m_k is the number of cycles of length k in the cycle representation of the permutation π.

■ For $\mathcal{S}_4$ there are five cycle types

$$(1^4), (1^2\ 2^1), (2^2), (1^1\ 3^1), (4^1) \tag{2.5}$$

Normally exponents of unity are omitted and Eq.(2.5) written as

$$(1^4), (1^2 2), (2^2), (13), (4) \tag{2.6}$$

■ Cycle types may be equally well labelled by ordered partitions of the integer n

$$\lambda = (\lambda_1 \lambda_2 \ldots \lambda_\ell) \tag{2.6}$$

where the λ_i are weakly decreasing and

$$\sum_{i=1}^{\ell} \lambda_i = n \tag{2.7}$$

The partition is said to be of *length* ℓ_λ and of *weight* $w_\lambda = n$. In terms of partitions the cycle types for $\mathcal{S}_5$ are

$$(1^5), (21^3), (2^2 1), (32), (31^2), (41), (5) \tag{2.8}$$

■ 2.3 Conjugacy Classes of $\mathcal{S}_n$

In any group G the elements g and h are *conjugates* if

$$g = khk^{-1} \qquad \text{for some} \qquad k \in G \tag{2.9}$$

The set of all elements conjugate to a given g is called the *conjugacy class* of g which we denote as K_g.

■ Exercises

2.3 Show that for $\mathcal{S}_4$ there are five conjugacy classes that may be labelled by the five partitions of the integer 4.

2.4 Show that the permutations, expressed in cycles, with cycles of length one suppressed, divide among the conjugacy classes as

$$\begin{aligned}(1^4) &\supset e \\ (21^2) &\supset (12), (13), (14), (23), (24), (34) \\ (2^2) &\supset (12)(34), (13)(24), (14)(23) \\ (31) &\supset (123), (124), (132), (134), (142) \\ &\quad (143), (234), (243) \\ (4) &\supset (1234), (1243), (1342), (1432)\end{aligned} \tag{2.10}$$

In general two permutations are in the same conjugacy class if, and only if, they are of the same cycle type. The number of classes of $\mathcal{S}_n$ is equal the number of partitions of the integer n.

If $\lambda = (1^{m_1}2^{m_2}\dots n^{m_n})$ then the number of permutations k_λ in the class (λ) of $\mathcal{S}_n$ is

$$k_\lambda = \frac{n!}{1^{m_1}m_1!2^{m_2}m_2!\dots n^{m_n}m_n!} \tag{2.11}$$

■ 2.4 The Cayley Table for $\mathcal{S}_3$

	e	(12)	(13)	(23)	(132)	(123)
e	e	(12)	(13)	(23)	(132)	(123)
(12)	(12)	e	(132)	(123)	(13)	(23)
(13)	(13)	(123)	e	(132)	(23)	(12)
(23)	(23)	(132)	(123)	e	(12)	(13)
(132)	(132)	(23)	(12)	(13)	(123)	e
(123)	(123)	(13)	(23)	(12)	e	(132)

■ 2.5 Transpositions and cycles of $\mathcal{S}_n$

1. A cycle of order two is termed a *transposition*.
2. A transposition $(i,\, i+1)$ is termed an *adjacent transposition*.
3. The entire symmetric group $\mathcal{S}_n$ can be generated (or given a *presentation* in terms of the set of adjacent transpositions

$$(1\,2),\ (2\,3),\ \dots,\ (n-1\,n) \tag{2.12}$$

■ If $\pi = \tau_1\tau_2\dots\tau_k$, where the τ_i are transpositions then the *sign* of π is defined to be

$$sgn(\pi) = (-1)^k \tag{2.13}$$

If the number of cycles of *even* order is *even* then the permutation is *even* or *positive*; if it is *odd* then the permutation is *odd* or *negative*.

■ 2.6 The Presentation of $\mathcal{S}_n$

Let us designate an adjacent transposition by

$$s_i = (i, i+1) \qquad \text{for} \quad i = 1, 2, \dots, n-1 \tag{2.14}$$

then we can give a *presentation* of the symmetric group $\mathcal{S}_n$ in terms of the s_i via the following three relations:-

$$s_i^2 = 1 \qquad \text{for} \quad i = 1, 2, \dots, n-1 \tag{2.15a}$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \qquad \text{for} \quad i = 1, 2, \dots, n-2 \tag{2.15b}$$

$$s_i s_j = s_j s_i \qquad \text{for} \quad |i-j| \geq 2 \tag{2.15c}$$

Every permutation π in S_n can be expressed as a *reduced word* of minimal length $\ell(\pi)$ in the s_i.

■ Exercise

2.5 Verify the last sentence in the case of $\mathcal{S}_3$

■ 2.7 Note on Hecke algebra $\mathcal{H}_n(q)$ of type $\mathcal{A}_{n-1}$

We can $q-$deform the presentation of $\mathcal{S}_n$ to give the complex Hecke algebra $\mathcal{H}_n(q)$, with q an arbitrary but fixed complex parameter, generated by g_i with $i = 1, 2, \dots, n-1$ subject to the relations:

$$g_i^2 = (q-1)g_i + q \qquad \text{for} \quad i = 1, 2, \dots, n-1 \tag{2.16a}$$

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \qquad \text{for} \quad i = 1, 2, \dots, n-2 \tag{2.16b}$$

$$g_i g_j = g_j g_i \qquad \text{for} \quad |i-j| \geq 2 \tag{2.16c}$$

For $q = 1$ these relations are exactly those appropriate to the symmetric group $\mathcal{S}_n$. There exists a map h from $\mathcal{S}_n$ to $\mathcal{H}_n(q)$ such that $h(s_i) = g_i$ and $h(\pi) = g_{i_1}g_{i_2}\dots g_{i_m}$ for any permutation $\pi = s_{i_1}s_{i_2}\dots s_{i_m} \in$

$\mathcal{S}_n$. The set of reduced words $h(\pi)$ for all $n!$ permutations $\pi \in \mathcal{S}_n$ forms a basis of $\mathcal{H}_n(q)$. For more details see:- R. C. King and B. G. Wybourne, *J. Phys. A: Math. Gen. 23* L1193 (1990).

■ 2.8 The Alternating Group $\mathcal{A}_n$

The set of *even* permutations form a subgroup of $\mathcal{S}_n$ known as the *alternating group* $\mathcal{A}_n$ and has precisely half the elements of $\mathcal{S}_n$ i.e. $(\frac{1}{2})n!$.

■ Exercises

2.6 Show that the set of six matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \tag{3.17}$$

with the usual rule of matrix multiplication form a group isomorphic to $\mathcal{S}_3$.

2.7 Show that the symmetric group $\mathcal{S}_n$ has two one-dimensional representations, a symmetric representation where every element is mapped onto
unity and an antisymmetric representation where the elements are mapped onto the sign defined in Eq. (2.13).

Symmetric Functions and the Symmetric Group 3

B. G. Wybourne

For every complex question there is a simple answer
– and it's wrong.
– *H. L. Mencken*

- 3.1 Semistandard numbering and Young tableaux
- Many different prescriptions can be given for injecting numbers into the boxes of a frame.
- The standard numbering is intimately associated with the symmetric group $\mathcal{S}_n$.
- Another important numbering prescription is that of *semistandard* numbering where now numbers 1, 2, ..., d are injected into the boxes of a frame F^{λ} such that:
  - i. Numbers are non-decreasing across a row going from left to right.
  - ii. Numbers are positively increasing in columns from top to bottom.
- The first condition permits repetitions of integers.

Using the numbers 1, 2, 3 in the frame $F^{2\,1}$ we obtain the 8 tableaux

$$\begin{array}{ccccc} \begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array} & & \\ \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 1&3\\\hline 3\\\cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 2&3\\\hline 3\\\cline{1-1}\end{array} \end{array} \tag{3.1}$$

Had we chosen $d = 2$ we would have obtained just two tableaux while $d = 4$ yields twenty tableaux. In general, for a frame F^{λ} a semistandard numbering using the integers 1, 2, ..., d leads to

$$f_d^{\lambda} = \frac{G_d^{\lambda}}{H_{\lambda}} \tag{3.2}$$

where H_{λ} is the product of the hook lengths h_{ij} of the frame and

$$G_d^{\lambda} = \prod_{(i,j)\in\lambda} (d+i-j) \tag{3.3}$$

Thus for $d = 5$ and $\lambda = (4\,2\,1)$ we have $H_{(4\,2\,1)} = 144$ and $G_5^{\{4\,2\,1\}} = 100800$ from which we deduce that

$$f_5^{\{4\,2\,1\}} = 700$$

which is the dimension of the irreducible representation $\{4\,2\,1\}$ of the general linear group $GL(5)$.

- In general, f_d^{λ} is the dimension of the irreducible representation $\{\lambda\}$ of $GL(d)$. Since the representations of $GL(d)$ labelled by partitions λ remain irreducible under restriction to the unitary group $U(d)$ Eq.(3.3) is valid for computing the dimensions of the irreducible representations of the unitary group $U(d)$.
- The same rules for a semistandard numbering may be applied to the skew frames $F^{\lambda/\mu}$. Thus for $F^{542/21}$ an allowed semistandard numbering using just the integers 1 and 2 would be

$$\begin{array}{ccccc} & & \boxed{1} & \boxed{1} & \boxed{1} \\ & \boxed{1} & \boxed{2} & \boxed{2} & \\ \boxed{1} & \boxed{2} & & & \end{array}$$

- Note that our semistandard numbering yields what in the mathematical literature are commonly referred to as *semistandard* Young tableaux. Other numberings are possible and have been developed for all the classical Lie algebras.

■ Exercises

3.1 Draw the frames $F^{2^2/1}$, $F^{43^21/421^2}$, and $F^{321/21}$.

3.2 Use the integers 1, 2, 3 to construct the complete set of semistandard tableaux for the frame $F^{43^21/421^2}$ and show that the same number of tableaux arise for the frame F^{21}.

3.3 Make a similar semistandard numbering for the frame $F^{321/21}$ and show that the same number of semistandard tableaux arise in the set of frames $F^{3} + 2F^{21} + F^{1^3}$.

■ 3.2 Young tableaux and monomials

A numbered frame may be associated with a unique monomial by replacing each integer i by a variable x_i. Thus the Young tableau

1	1	2	4	5
3	3	3	5	
4	6	7		
5	7	8		
6	8			
7				

can be associated with the monomial $x_1^2\, x_2\, x_3^3\, x_4^2\, x_5^3\, x_6^2\, x_7^3\, x_8^2$

■ 3.3 Monomial symmetric functions

Consider a set of variables $(x) = x_1, x_2, \ldots, x_d$. A *symmetric* monomial

$$\boxed{m_\lambda(x) = \sum_\alpha x^\alpha} \tag{3.4}$$

involves a sum over all distinct permutations α of $(\lambda) = (\lambda_1, \lambda_2, \ldots)$. Thus if $(x) = (x_1, x_2, x_3)$ then

$$m_{21}(x) = x_1^2\, x_2 + x_1^2\, x_3 + x_1\, x_2^2 + x_1\, x_3^2 + x_2^2\, x_3$$
$$m_{1^3}(x) = x_1\, x_2\, x_3$$

The semistandard numbering of $(\lambda) = (2\,1)$ with 1, 2, 3 corresponds to the sum of monomials

$$\boxed{m_{21}(x) + 2m_{1^3}(x)} \tag{3.5}$$

The same linear combination occurs for any number of variables with $d \geq 3$.

The monomials $m_\lambda(x)$ are *symmetric functions.* If $\lambda \vdash n$ then $m_\lambda(x)$ is homogeneous of degree n. Unless otherwise stated we shall henceforth assume that x involves an infinite number of variables x_i.

The *ring of symmetric functions* $\Lambda = \Lambda(x)$ is the vector space spanned by all the $m_\lambda(x)$. This space can be decomposed as

$$\Lambda = \oplus_{n \geq 0} \Lambda^n \tag{3.6}$$

where Λ^n is the space spanned by all m_λ of degree n. Thus the $\{m_\lambda | \lambda \vdash n\}$ form a basis for the space Λ^n which is of dimension $p(n)$ where $p(n)$ is the number of partitions of n. It is of interest to ask if other bases can be constructed for the space Λ^n.

■ 3.4 The classical symmetric functions

Three other classical bases are well-known - some since the time of Newton.

1. The elementary symmetric functions
   The $n-$th elementary symmetric function e_n is the sum over all products of n distinct variables x_i, with $e_0 = 1$ and generally

$$e_n = m_{1^n} = \sum_{i_1 < i_2 \ldots < i_n} x_{i_1}\, x_{i_2}\, \ldots\, x_{i_n} \tag{3.7}$$

The *generating function* for the e_n is

$$E(t) = \sum_{n\geq 0} e_n t^n = \prod_{i\geq 1}(1+x_i t) \tag{3.8}$$

2. The complete symmetric functions
   The $n-$th complete or *homogeneous* symmetric function h_n is the sum of all monomials of total degree n in the variables $x_1, x_2, \ldots$, with $h_0 = 1$ and generally

$$h_n = \sum_{|\lambda|=n} m_\lambda = \sum_{i_1 \leq i_2 \ldots \leq i_n} x_{i_1} x_{i_2} \ldots x_{i_n} \tag{3.9}$$

   The generating function for the h_n is

$$H(t) = \sum_{n\geq 0} h_n t^n = \prod_{i\geq 1}(1-x_i t)^{-1} \tag{3.10}$$

3. The power sum symmetric function
   The $n-$th power sum symmetric function is

$$p_n = m_n = \sum_{i\geq 1} x_i^n \tag{3.11}$$

   The generating function for the p_n is

$$\begin{aligned} P(t) = \sum_{n\geq 1} p_n t^{n-1} &= \sum_{i\geq 1}\sum_{n\geq 1} x_i^n t^{n-1} \\ &= \sum_{i\geq 1} \frac{x_i}{1-x_i t} \\ &= \sum_{i\geq 1} \frac{d}{dt} \log \frac{1}{1-x_i t} \end{aligned} \tag{3.12}$$

   and hence

$$\begin{aligned} P(t) &= \frac{d}{dt} \log \prod_{i\geq 1}(1-x_i t)^{-1} \\ &= \frac{d}{dt} \log H(t) \\ &= H^{'}(t)/H(t) \end{aligned} \tag{3.13}$$

   Similarly,

$$P(-t) = \frac{d}{dt} \log E(t) = E^{'}(t)/E(t) \tag{3.14}$$

   Equation (3.13) leads to the relationship

$$nh_n = \sum_{r=1}^{n} p_r \, h_{n-r} \tag{3.15}$$

   It follows from (3.13) that

$$\begin{aligned} H(t) &= \exp \sum_{n\geq 1} p_n \, t^n/n \\ &= \prod_{n\geq 1} \exp(p_n \, t^n/n) \\ &= \prod_{n\geq 1} \sum_{m_n=0}^{\infty} (p_n \, t^n)^{m_n}/n^{m_n}.m_n! \end{aligned} \tag{3.15}$$

and hence

$$H(t) = \sum_{\lambda} z_{\lambda}^{-1} p_{\lambda}\, t^{|\lambda|} \tag{3.16}$$

where

$$z_{\lambda} = \prod_{i \geq 1} i^{m_i}.m_i! \tag{3.17}$$

where $m_i = m_i(\lambda)$ is the number of parts of λ equal to i.
Defining

$$\varepsilon_{\lambda} = (-1)^{|\lambda| - \ell(\lambda)} \tag{3.18}$$

we can show in an exactly similar manner to that of Eq.(3.16) that

$$E(t) = \sum_{\lambda} \varepsilon_{\lambda} z_{\lambda}^{-1} p_{\lambda}\, t^{|\lambda|} \tag{3.19}$$

It then follows from Eqs.(3.16) and (3.19) that

$$h_n = \sum_{|\lambda|=n} z_{\lambda}^{-1} p_{\lambda} \tag{3.20}$$

and

$$e_n = \sum_{|\lambda|=n} \varepsilon_{\lambda} z_{\lambda}^{-1} p_{\lambda} \tag{3.21}$$

■ Exercises

3.4 Show that for $n = 3$

$$\begin{aligned} p_3 &= x_1^3 + x_2^3 + x_3^3 + \ldots \\ e_3 &= x_1\, x_2\, x_3 + x_1\, x_2\, x_4 + x_2\, x_3\, x_4 + \ldots \\ h_3 &= x_1^3 + x_2^3 + \ldots + x_1^2\, x_2 + x_1\, x_2^2 + \ldots + x_1\, x_2\, x_3 + x_1\, x_2\, x_4 + \ldots \end{aligned} \tag{3.22}$$

3.5 Noting Eqs. (3.8) and (3.10) and that $H(t)E(-t) = 1$, show that

$$\sum_{r=0}^{n} (-1)^r h_{n-r}\, e_r = 0 \tag{3.23}$$

for $n \geq 1$.

3.6 Use Eq.(3.15) to show that

$$e_n = det(h_{1-i+j})_{1 \leq i,j \leq n} \tag{3.24}$$

and hence

$$h_n = det(e_{1-i+j})_{1 \leq i,j \leq n} \tag{3.25}$$

3.7 Use Eq.(3.15) to obtain the determinantal expressions

$$p_n = \begin{vmatrix} e_1 & 1 & 0 & \ldots & 0 \\ 2e_2 & e_1 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ ne_n & e_{n-1} & e_{n-2} & \ldots & e_1 \end{vmatrix} \tag{3.26}$$

$$n!e_n = \begin{vmatrix} p_1 & 1 & 0 & \ldots & 0 \\ p_2 & p_1 & 2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ p_{n-1} & p_{n-2} & . & \ldots & n-1 \\ p_n & p_{n-1} & . & \ldots & p_1 \end{vmatrix} \tag{3.27}$$

$$(-1)^{n-1}p_n = \begin{vmatrix} h_1 & 1 & 0 & \dots & 0 \\ 2h_2 & h_1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ nh_n & h_{n-1} & h_{n-2} & \dots & h_1 \end{vmatrix} \tag{3.28}$$

$$n!h_n = \begin{vmatrix} p_1 & -1 & 0 & \dots & 0 \\ p_2 & p_1 & -2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ p_{n-1} & p_{n-2} & . & \dots & -n+1 \\ p_n & p_{n-1} & . & \dots & p_1 \end{vmatrix} \tag{3.29}$$

■ 3.5 Multiplicative bases for Λ^n

The three types of symmetric functions, h_n, e_n, p_n, do not have enough elements to form a basis for Λ^n, there must be one function for every partition $\lambda \vdash n$. To that end in each case we form *multiplicative* functions f_λ so that for each $\lambda \vdash n$

$$f_\lambda = f_{\lambda_1} f_{\lambda_2} \dots f_{\lambda_\ell} \tag{3.30}$$

where $f = e,\, h,$ or p Thus, for example,

$$e_{21} = e_2 \cdot e_1 = (x_1 x_2 + x_1 x_3 + x_2 x_3 + \dots)(x_1 + x_2 + x_3 + \dots)$$

■ 3.6 The Schur functions

The symmetric functions

$$m_\lambda,\, e_\lambda,\, h_\lambda,\, p_\lambda \tag{3.31}$$

where $\lambda \vdash n$ each form a basis for Λ^n. A very important fifth basis is realised in terms of the Schur functions, s_λ, *or for brevity, $S-$functions* which may be variously defined. Combinatorially they may be defined as

$$s_\lambda(x) = \sum_T x^T \tag{3.32}$$

where the summation is over all semistandard
$\lambda-$tableaux T. For example, consider the $S-$functions s_λ in just three variables $(x_1,\, x_2,\, x_3)$. For $\lambda = (2\,1)$ we have the eight tableaux T found earlier

$$\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1&3\\\hline 3\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2&3\\\hline 3\\\cline{1-1}\end{array} \tag{3.33}$$

Each tableaux T corresponds to a monomial x^T to give

$$\begin{aligned} s_{2\,1}(x_1,\, x_2,\, x_3) =& x_1^2 x_2 + x_1^2 x^3 + x_1 x_2^2 + x_1 x_2 x_3 + x_1 x_2 x_3 + x_1 x_3^2 \\ &+ x_2^2 x_3 + x_2 x_3^2 \end{aligned} \tag{3.34}$$

We note that the monomials in Eq.(3.34) can be expressed in terms of just two *symmetric monomials* in the three variables $(x_1,\, x_2,\, x_3)$ to give

$$s_{2\,1}(x_1,\, x_2,\, x_3) = m_{2\,1}(x_1,\, x_2,\, x_3) + 2m_{1^3}(x_1,\, x_2,\, x_3) \tag{3.35}$$

In an arbitrary number of variables

$$s_{2\,1}(x) = m_{2\,1}(x) + 2m_{1^3}(x) \tag{3.36}$$

This is an example of the general result that the
$S-$function may be expressed as a linear combination of symmetric monomials as indeed would be expected if the $S-$functions are a basis of Λ^n. In fact

$$s_\lambda(x) = \sum_{\mu \vdash n} K_{\lambda\mu} m_\mu \tag{3.37}$$

where $|\lambda| = n$ and $K_{\lambda\lambda} = 1$. The $K_{\lambda\mu}$ are the elements of an upper triangular matrix K known as the Kostka matrix. K is an example of a *transition matrix* that relates one symmetric function basis to another.

■ 3.7 Calculation of the elements of the Kostka matrix

The elements $K_{\lambda\mu}$ of the Kostka matrix may be readily calculated by the following algorithm :

i. Draw the frame F^λ.
ii. Form all possible semistandard tableaux that arise in numbering F^λ with μ_1 ones, μ_2 twos etc.
iii. $K_{\lambda\mu}$ is the number of semistandard tableaux so formed.

Thus calculating $K_{(42)\,(2^2\,1^2)}$ we obtain the four semistandard tableaux

$$\begin{array}{cccc}1122 & 1123 & 1124 & 1134\\ 34 & 24 & 23 & 22\end{array}$$

and hence $K_{(42)\,(2^2\,1^2)} = 4$.

■ Exercises

3.8 Construct the Kostka matrix for $\lambda, \mu \vdash 4$.

3.9 Show that in the variables $(x_1,\, x_2,\, x_3)$ the evaluation of the determinantal ratio

$$\frac{\begin{vmatrix} x_1^4 & x_1^2 & 1\\ x_2^4 & x_2^2 & 1\\ x_3^4 & x_3^2 & 1\end{vmatrix}}{\begin{vmatrix} x_1^2 & x_1 & 1\\ x_2^2 & x_2 & 1\\ x_3^2 & x_3 & 1\end{vmatrix}}$$

yields the monomial content of the $S-$function s_{21} in three variables as found in Eq.(3.36). N.B. The above exercise is tedious by hand but trivial using MAPLE.

The last exercise is an example of the classical definition, as opposed to the equivalent combinatorial definition given in Eq.(3.32), given first by Jacobi, namely,

$$s_\lambda = s_\lambda(x_1,\, x_2,\, \ldots, x_n) = \frac{a_{\lambda+\delta}}{a_\delta} \tag{3.38}$$

where λ is a partition of length $\leq n$ and
$\delta = (n-1,\, n-2,\, \ldots,\, 1,\, 0)$ with

$$a_{\lambda+\delta} = det(x_i^{\lambda_j+n-j})_{1\leq i,j\leq n} \tag{3.39}$$

and

$$a_\delta = \prod_{1\leq i,j\leq n} (x_i - x_j) = det(x_i^{n-j}) \tag{3.40}$$

is the *Vandermonde determinant.* Note that the Vandermonde determinant is an *alternating* or *antisymmetric* function. Any *even* power of the Vandermonde determinant is an *symmetric* function. This has important applications in the interpretation of the quantum Hall effect.

■ 3.8 Non-standard $S-$functions

The $S-$functions are symmetric functions indexed by ordered partitions λ. We shall frequently write $S-$functions $s_\lambda(x)$ as $\{\lambda\}(x)$ or, since we will generally consider the number of variables to be unrestricted, just $\{\lambda\}$. As a matter of notation the partitions will normally be written without spacing or commas separating the parts where $\lambda_i \leq 9$. A space will be left after any part $\lambda_i \geq 10$. Thus we write $\{12, 11, 9, 8, 3, 2, 1\} \equiv \{12\ \ 11\ \ 98321\}$ While we have defined the $S-$function in terms of ordered partitions we sometimes encounter $S-$functions that are not in the standard form and must convert such *non-standard* $S-$functions into standard $S-$functions. Inspection of the determinantal forms of the $S-$function leads to the establishment of the following *modification rules* :

$$\{\lambda_1, \lambda_2, \ldots, -\lambda_\ell\} = 0 \tag{3.41}$$

$$\{\lambda_1,\ldots,\lambda_i,\lambda_{i+1},\ldots,\lambda_\ell\} = -\{\lambda_1,\ldots,\lambda_{i+1}-1,\lambda_i+1,\ldots,\lambda_\ell\} \tag{3.42}$$

$$\{\lambda\} = 0 \qquad \text{if } \lambda_{i+1} = \lambda_i + 1 \tag{3.43}$$

Repeated application of the above three rules will reduce any non-standard $S-$function to either zero or to a signed standard $S-$function. In the process of using the above rules trailing zero parts are omitted

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■ Exercise

3.10 Show that

$$\{24\} = -\{3^2\}, \quad \{141\} = -\{321\}$$
$$\{14-25-14\} = -\{3^3 2\}$$
$$\{3042\} = 0, \quad \{3043\} = \{3^2 2\}$$

■ 3.9 Skew $S-$functions

The combinatorial definition given for $S-$functions in Eq.(3.32) is equally valid for skew tableaux and can hence be used to define *skew* $S-$functions $s_{\lambda/\mu}(x)$ or $\{\lambda/\mu\}$. Since the $s_{\lambda/\mu}(x)$ are symmetric functions they must be expressible in terms of $S-$functions $s_\nu(x)$ such that

$$s_{\lambda/\mu} = \sum_\nu c^\lambda_{\mu\nu} s_\nu \tag{3.44}$$

It may be shown that the coefficients $c^\lambda_{\mu\nu}$ are necessarily non-negative integers and symmetric with respect to μ and ν. The coefficients $c^\lambda_{\mu\nu}$ are commonly referred to as *Littlewood-Richardson* coefficients.

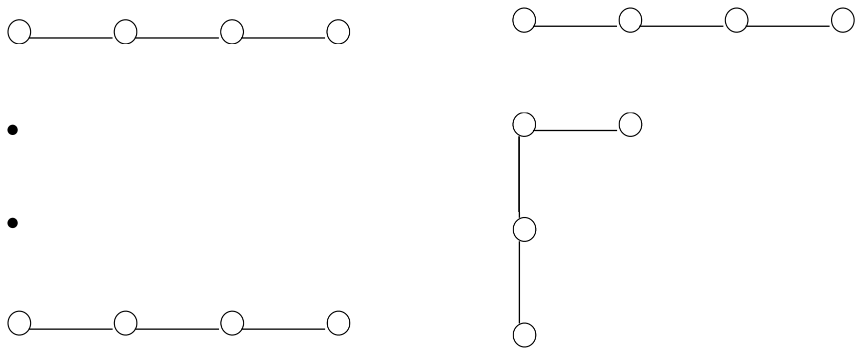
■ 3.10 Slinkies and Modification Rules

In situations involving extensive use of modification rules and in particular when one is trying to derive general formulae the use of *slinkies* can be very useful (KWY:King, Wybourne and Yang, *J. Phys. A: Math. Gen.* **22**, 4519 (1989)). (see also Chen, Garsia and Remmel, *Contemp. Math.* **34**, 109 (1984)). A slinky of length q is a diagram of q circles joined by $q-1$ links. A slinky can be folded so as to take the shape of a continuous boundary strip of a regular Young diagram, with each of the links eithehorizontal or vertical and its circles forming part of the boundary of such a diagram. The *sign* of the slinky is defined to be $(-1)^{r-1}$ where r is the number of rows occupied by the circles of the slinky, so that $r-1$ is the number of vertical links of the slinky.

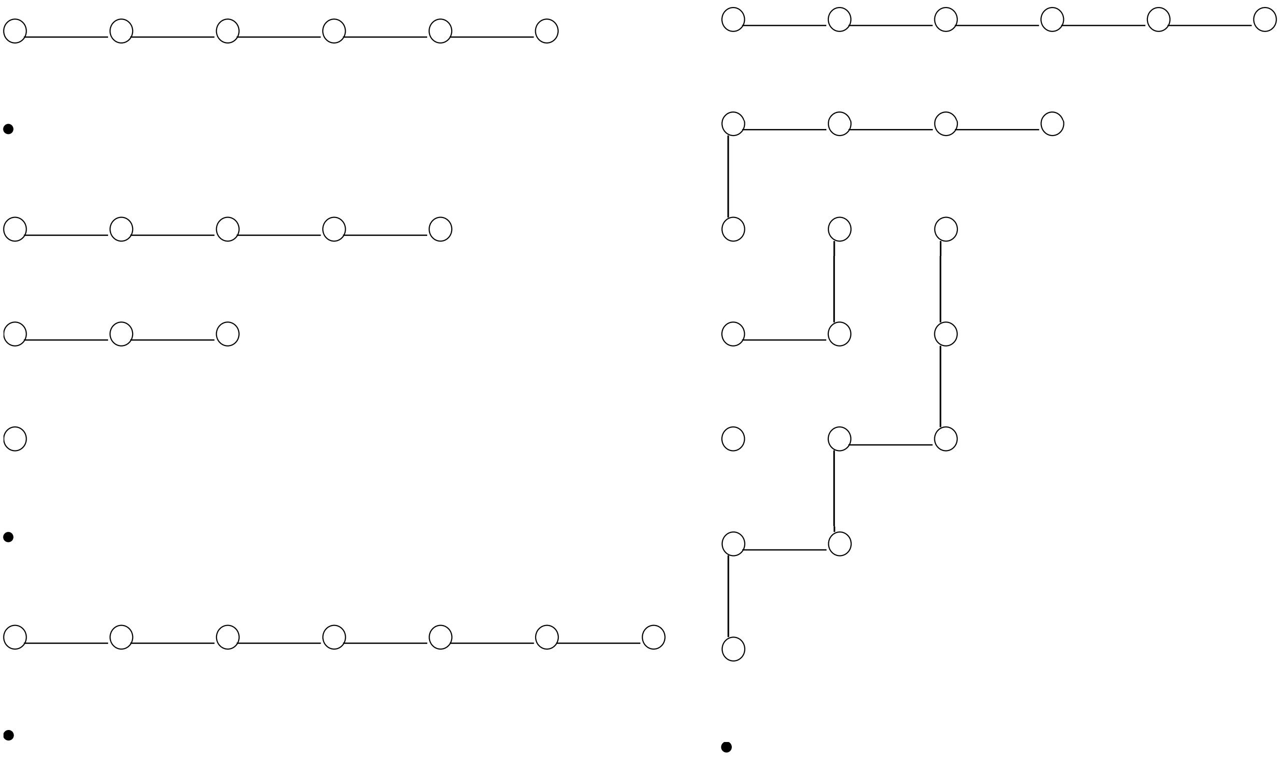
The modification rules for non-standard $S-$functions can be implemented in terms of folding operations of the slinkies that make up the Young diagram as follows:

1. Draw the slinky diagram corresponding to the non-standard $S-$function $\{\lambda_1,\lambda_2,\ldots,\lambda_p\}$.
2. Successively, for $i = 1,2,\ldots,p$, while holding the starting positions of the slinkies fixed, fold (if necessary) the $i-$th slinky of length λ_i into the shape of the unique standard continuous boundary strip such that the first i rows of the resulting diagram constitute a regular Young diagram. If this is not possible then $\{\lambda\} = 0$. Otherwise we obtain, after folding the last slinky, the regular Young diagram corresponding to some standard $S-$function $\{\mu\}$. The final result is then $\{\lambda\} = (-1)^v\{\mu\}$ where v is the total number of vertical links in the diagram.

We illustrate the application of the method of slinkies with two examples.



$\{4004\} \quad \Rightarrow \quad \{4211\}$



$\{60531070\} \quad \Rightarrow \quad \{64333210\}$

The principal application of the slinky method is to the expansion of symmetric generating functions as a sum of $S-$functions. Thus, for example, one (KWY) can show that

$$\prod_i (1 + x_i - xi^2) = \sum_{q,r=0}^{\infty} (-1)^q f_{r+1} \{2^q 1^r\}$$

where f_{r+1} is the $(r+1)-$th Fibonacci number.

■ Exercises

3.11 Show that

$$\{24\} = -\{3^2\}, \quad \{141\} = -\{321\}, \quad \{3042\} = 0, \quad \{3043\} = +\{3^2 2\}, \quad \{14-25-14\} = -\{3^3 2\}$$

3.12 Extend the slinky algorithm to include the possibility of negative parts and then show that $\{14-25-14\} = -\{3^3 2\}$.

3.13 Use the method of slinkies to show that

$$\{60531070\} = \{643^3 21\} \quad \text{and} \quad \{61131090\} = 0$$

■ General Remarks concerning $S-$functions

The $S-$functions are symmetric functions and form an integral basis for the ring of symmetric functions and hence may be expressed in terms of the classical symmetric functions $e_\lambda,\ h_\lambda,\ m_\lambda,\ f_\lambda$. Transition matrices can be defined for taking one from members of one basis to another. The transition matrices can be expressed in terms of the Kostka matrix $K_{\lambda\mu}$ and the transposition matrix

$$J_{\lambda\mu} = \begin{cases} 1, & \text{if} \tilde{\lambda} = \mu; \\ 0. & \text{otherwise} \end{cases} \tag{59}$$

The relevant transition matrices are tabulated in Macdonald (p56). These matrices all involve integers only.

The elementary and homogeneous symmetric functions e_n and h_n are special cases of $S-$functions , namely,

$$e_n \equiv \{1^n\} \qquad h_n \equiv \{n\} \tag{3.45}$$

$S-$functions arise in many situations. In the next few lectures we shall explore some of their properties that are relevant to applications in physics an chemistry. To proceed to these we must first consider the Littlewood-Richardson rule and then discuss the role of $S-$functions in the character theory of the symmetric group $S(n)$ and the unitary group $U(n)$.

Symmetric Functions and the Symmetric Group 4

B. G. Wybourne

'Fred!' cried Mr Swiveller, tapping his nose, 'a word to the wise is sufficient for them - we may be good and happy without riches, Fred.'
Charles Dickens *Old Curiosity Shop* (1841).

■ 4.1 The Littlewood-Richardson rule

The product of two $S-$functions can be written as a sum of $S-$functions, viz.

$$s_\mu . s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda \tag{4.1}$$

The Littlewood-Richardson coefficients $c^\lambda_{\mu\nu}$ in Eqs. (3.44) for skew $S-$function and (4.1) are identical, though the summations are of course different. In both cases $|\mu| + |\nu| = |\lambda|$. A rule for evaluating the coefficients $c^\lambda_{\mu\nu}$ was given by Littlewood and Richardson in 1934 and has played a major role in all subsequent developments. The rule may be stated in various ways. We shall state it first in terms of semistandard tableaux and then also give the rule for evaluating the product given in Eq.(4.1) which is commonly referred to as the *outer multiplication* of $S-$functions. In each statement the concepts of a *row-word* and of a *lattice permutation* is used.

■ 4.2 Definition 1 A word

Let T be a tableau. From T we derive a row-word or sequence $w(T)$ by reading the symbols in T from right to left (i.e. as in Arabic or Hebrew) in successive rows starting at the top row and proceeding to the bottom row

Thus for the tableau

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 3 \\ \hline 2 & 2 & 3 & 3 \\ \cline{1-4} 4 & 4 \\ \cline{1-2} 5 & 6 \\ \cline{1-2} 7 \\ \cline{1-1} 8 \\ \cline{1-1} \end{array}$$

we have the word $w(T) = 322113322446578$ and for the skew tableau

$$\begin{array}{ccccc} & & \boxed{1} & \boxed{1} & \boxed{1} \\ & \boxed{1} & \boxed{2} & \boxed{2} & \\ \boxed{1} & \boxed{2} & & & \end{array}$$

we have the word $w(T) = 11122121$.

■ 4.3 Definition 2 A lattice permutation

A word $w = a_1 a_2 \ldots a_N$ in the symbols $1, 2, \ldots, n$ is said to be a lattice permutation if for $1 \leq r \leq N$ and $1 \leq i \leq n-1$, the number of occurrences of the symbol i in $a_1 a_2 \ldots a_r$ is not less than the number of occurrences of $i+1$.

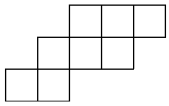
Thus the word $w(T) = 322113322446578$ is clearly not a lattice permutation whereas the word $w(T) = 11122121$ is a lattice permutation. The word $w(T) = 12122111$ is not a lattice permutation since the sub-word 12122 has more twos than ones.

■ Theorem 1 *The value of the coefficient $c^\lambda_{\mu\nu}$ is equal to the number of semistandard tableaux T of shape $F^{\lambda/\mu}$ and content ν such that $w(T)$ is a lattice permutation.*

By *content* ν we mean that each tableau T contains ν_1 ones, ν_2 twos, etc.

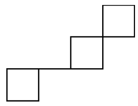
■ Example

Let us evaluate the coefficient $c^{\{542\}}_{\{431\}\{21\}}$. We first draw the frame $F^{\{542/21\}}$.



Into this frame we must inject the content of $\{431\}$ i.e. 4 ones, 3 twos and 1 three in such a way that we have a lattice permutation. We find two such numberings

and hence $c^{\{542\}}_{\{431\}\{21\}} = 2$. Note that in the evaluation we had a choice, we could have, and indeed more simply, evaluated $c^{\{542\}}_{\{21\}\{431\}}$. In that case we would have drawn the frame $F^{\{542/431\}}$ to get



Note that in this case the three boxes are disjoint. This skew frame is to be numbered with two ones and one 2 leading to the two tableaux

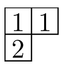
verifying the previous result. Theorem 1 gives a direct method for evaluating the Littlewood-Richardson coefficients. These coefficients can be used to evaluate both skews and products. It is sometimes useful to state a procedure for directly evaluating products.

■ Theorem 2 *to evaluate the* $S-$*function product* $\{\mu\}.\{\nu\}$

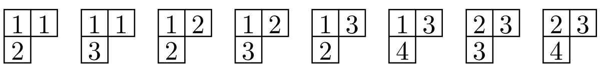
1. *Draw the frame* F^{μ} *and place* ν_1 *ones in the first row,* ν_2 *twos in the second row etc until the frame is filled with integers.*
2. *Draw the frame* F^{ν} *and inject positive integers to form a semistandard tableau such that the word formed by reading from right to left starting at the top row of the first frame and moving downwards along successive rows to the bottom row and then continuing through the second frame is a lattice permutation.*
3. *Repeat the above process until no further words can be constructed.*
4. *Each word corresponds to an* $S-$*function* $\{\lambda\}$ *where* λ_1 *is the number of ones,* λ_2 *the number of twos etc.*

As an example consider the $S-$function product $\{21\} \cdot \{21\}$.

Step 1 gives the tableau



Steps 2 and 3 lead to the eight numbered frames



Step 4 then lead to the eight words

112112	112113	112212	112213
112312	112314	112323	112324

from which we conclude that

$$\{21\}.\{21\} = \{42\} + \{41^2\} + \{3^2\} + 2\{321\} + \{31^3\} + \{2^3\} + \{2^21^2\}$$

I have made only one non-mathematical discovery in my life, the discovery of the exclusion principle; and that was what I was given the Nobel prize for! (Wolfgang Pauli, 1956)

Dear Professor,
I must have a serious word with you today. Are you acquainted with a certain Mr. Schrödinger, who in the year 1922 (Zeits. fur Phys.,12) described a 'bemerkenswerte Eigenschaft der Quantebahnen'? Are you acquainted with this man? What! You affirm that you know him very well, that you were even present when he did this work and that you were his accomplice in it? That is absolutely unheard of. ......
With hearty greetings, I am
Yours very faithfully

Fritz London

■ 4.4 Relationship to the unitary group

We have explored various symmetric functions indexed by partitions and defined on sets of variables. The variables can admit many interpretations. In some instances we may choose a set of variables $1, q, q^2, \ldots, q^n$ (cf. Farmer, King and Wybourne, *J. Phys. A: Math. Gen.* **21**, 3979 (1988).) or we could even use a set of matrices. The link between $S-$functions and the character theory of groups is such that, if λ is a partition with $\ell(\lambda) \leq N$ and the eigenvalues of a group element, g, of the unitary group U_N are given by $x_j = \exp(i\phi_j)$ for $j = 1, 2, \ldots, N$ then the $S-$function

$$\begin{aligned}\{\lambda\} &= \{\lambda_1\lambda_2\ldots\lambda_N\} = s_\lambda(x)\\ &= s_\lambda(\exp(i\phi_1)\exp(i\phi_2)\ldots\exp(i\phi_N))\end{aligned} \tag{4.2}$$

is nothing other than the character of g in the irreducible representation of U_N conventionally designated by $\{\lambda\}$.

The Littlewood-Richardson rule gives the resolution of the Kronecker product $\{\mu\} \times \{\nu\}$ of U_N as

$$\{\mu\} \times \{\nu\} = \sum_{|\lambda|=|\mu|+|\nu|} c^{\{\lambda\}}_{\{\mu\}.\{\nu\}}\{\lambda\} \tag{4.3}$$

where the $c^{\{\lambda\}}_{\{\mu\}.\{\nu\}}$ are the usual Littlewood-Richardson coefficients. Equation (4.3) must be modified for partitions λ involving more than N parts. Here the *modification rule* is very simple. We simply discard all partitions involving more than N parts. We shall return to the unitary groups later.

4.5 Reduced notation for the symmetric group

The irreps of the symmetric group $S(N)$ are uniquely labelled by the partitions $\lambda \vdash N$, there being as many irreps of $S(N)$ as there are partitions of N. Consider the following Kronecker products in $S(N)$

$$\begin{aligned}\{21\}\circ\{21\} &= \{3\} + \{21\} + \{1^3\}\\ \{31\}\circ\{31\} &= \{4\} + \{31\} + \{2^2\} + \{21^2\}\\ \{41\}\circ\{41\} &= \{5\} + \{41\} + \{32\} + \{31^2\}\end{aligned}$$

It is apparent that the result stabilises at $N = 4$ and in general we could write

$$\{N-1,1\}\circ\{N-1,1\} = \{N,0\} + \{N-1,1\} + \{N-2,2\} + \{N-2,1^2\} \tag{4.4}$$

The above result would hold for all N provided we apply the modification rules to any non-standard $S-$functions. Thus

$$\begin{aligned}\{21\}\circ\{21\} &= \{3\} + \{21\} + \{12\} + \{1^3\}\\ &= \{3\} + \{21\} + \{1^3\}\end{aligned}$$

since $\{12\} = -\{12\} = 0$.

Equation (4.4) could be rewritten as

$$\langle 1\rangle \circ \langle 1\rangle = \langle 0\rangle + \langle 1\rangle + \langle 2\rangle + \langle 1^2\rangle \tag{4.5}$$

The above equation is an example of the use of *reduced notation* (cf. Scharf, Thibon and Wybourne, *J. Phys. A: Math. Gen.* **26, 7461 (1993)** (STW), Butler and King, *J. Math. Phys.* **14, 1176 (1973)**(BK) and references therein.) which makes use of the fact that the symmetric group is a subgroup of the general linear group $Gl(N)$. In the reduced notation the irrep label $\{\lambda\} = \{\lambda_1, \lambda_2, \ldots, \lambda_p\}$ in $S(N)$ is replaced by $\langle\lambda\rangle = \langle\lambda_2, \ldots, \lambda_p\rangle$. Given any irrep $\langle\mu\rangle$ in reduced notation it can be converted back into a standard irrep of $S(N)$ by prefixing it with a part $N - |\mu|$. For example, an irrep $\langle 21\rangle$ in reduced notation corresponds in $S(6)$ t0 $\{321\}$ or $\{921\}$ in $S(12)$. If $N - |\mu| \geq \mu_1$ then the irrep $\{N - |\mu|, \mu\}$ is assuredly a standard irrep of $S(N)$. However, if $N - |\mu| \langle \mu_1$ then the resulting irrep $\{N - |\mu|, \mu\}$ is non-standard and must be converted into standard form.

■ 4.5 Reduced Kronecker products for $S(N)$

BK have, following Littlewood, given the reduced Kronecker product as

$$\langle\lambda\rangle \circ \langle\mu\rangle = \sum_{\alpha,\beta,\gamma} \langle(\{\lambda\}/\{\alpha\}\{\beta\}) \cdot (\{\mu\}/\{\alpha\}\{\gamma\}) \cdot (\{\beta\} \circ \{\gamma\})\rangle \tag{4.6}$$

where the $\cdot$ signifies ordinary Littlewood-Richardson multiplication of the relevant $S-$function.

■ 4.6 Exercises

4.1 Show that $\langle 21\rangle \circ \langle 31\rangle$ evaluates as

$\langle 6\rangle$	$+ \langle 52\rangle$	$+ \langle 51^2\rangle$	$+ 4\langle 51\rangle$	$+ 3\langle 5\rangle$	$+ \langle 43\rangle$
$+ 2\langle 421\rangle$	$+ 6\langle 42\rangle$	$+ \langle 41^3\rangle$	$+ 6\langle 41^2\rangle$	$+ 10\langle 41\rangle$	$+ 5\langle 4\rangle$
$+ \langle 3^21\rangle$	$+ 3\langle 3^2\rangle$	$+ \langle 32^2\rangle$	$+ \langle 321^2\rangle$	$+ 8\langle 321\rangle$	$+ 11\langle 32\rangle$
$+ 4\langle 31^3\rangle$	$+ 12\langle 31^2\rangle$	$+ 13\langle 31\rangle$	$+ 5\langle 3\rangle$	$+ 2\langle 2^3\rangle$	$+ 3\langle 2^21^2\rangle$
$+ 9\langle 2^21\rangle$	$+ 8\langle 2^2\rangle$	$+ \langle 21^4\rangle$	$+ 6\langle 21^3\rangle$	$+ 11\langle 21^2\rangle$	$+ 9\langle 21\rangle$
$+ 3\langle 2\rangle$	$+ \langle 1^5\rangle$	$+ 3\langle 1^4\rangle$	$+ 4\langle 1^3\rangle$	$+ 3\langle 1^2\rangle$	$+ \langle 1\rangle$

4.2 Use the above result to deduce that for $S(5)$ $\{221\} \circ \{221\}$ evaluates as

$\{5\}$	$+ \{41\}$	$+ \{32\}$	$+ \{31^2\}$	$+ \{2^21\}$	$+ \{21^3\}$

4.3 Show that in $S(8)$ $\{521\} \circ \{431\}$ evaluates as

$\{71\}$	$+ 3\{62\}$	$+ 3\{61^2\}$	$+ 4\{53\}$	$+ 9\{521\}$	$+ 4\{51^3\}$
$+ 2\{4^2\}$	$+ 9\{431\}$	$+ 7\{42^2\}$	$+ 10\{421^2\}$	$+ 3\{41^4\}$	$+ 5\{3^22\}$
$+ 6\{3^21^2\}$	$+ 7\{32^21\}$	$+ 5\{321^3\}$	$+ \{31^5\}$	$+ \{2^4\}$	$+ 2\{2^31^2\}$
$+ \{2^21^4\}$					

■ 4.7 Kronecker products for two-row partitions

In quantum chemistry the Pauli exclusion principle restricts interest to irreps of $S(N)$ indexed by partitions into at most two parts. In terms of reduced notation two-row shapes become one-row shapes via the equivalence

$$\{N-k, k\} \circ \{N-\ell, \ell\} \equiv \langle k\rangle \circ \langle \ell\rangle \tag{4.7}$$

From Eq. (4.7) we are led directly to

$$\begin{aligned} \langle k\rangle \circ \langle \ell\rangle &= \sum_{p=0}^{min(k,\ell} \sum_{q=0}^{p} \langle\{k-p\} \cdot \{\ell - p\} \cdot \{p-q\}\rangle \\ &= \sum_{\lambda} c^{\nu}_{\lambda\mu} \langle\lambda\rangle \end{aligned} \tag{4.8}$$

The possible shapes for λ are severely constrained. The number of rows cannot exceed three. The multiplicity to be associated with a given shape λ can be readily determined by drawing the shape and then filling the cells, in accordance with the Littlewood-Richardson rule, with say $k - p$ circles $\circ$, $\ell - p$ stars $\circ$ and $p - q$ diamonds $\diamond$, where

$$k + \ell - p + q = \lambda_1 + \lambda_2, \ldots \tag{4.9}$$

Repeated cells will be marked with dots · . Consider the shape characterised by the one-row (m), the only case relevant to quantum chemistry. A typical filling is shown below:

○	○	·	·	·	○	○	○	·	·	·	○	◇	◇	·	·	·	◇

From which we can deduced that $c^{\langle m\rangle}_{\langle k\rangle\langle\ell\rangle}$ is the number of partitions of $k+\ell-m$ into two parts (p,q) with $p\geq q$ and $\ell\geq p$ leading to

$$c^{\langle m\rangle}_{\langle k\rangle\langle\ell\rangle} = \frac{1}{2}(\ell-k+m+2) \quad \textbf{for} \quad k>m \tag{4.9a}$$

$$c^{\langle m\rangle}_{\langle k\rangle\langle\ell\rangle} = \frac{1}{2}(k+\ell-m+2) \quad \textbf{for} \quad m\geq k \tag{4.9b}$$

and the coefficient symmetry

$$c^{\langle m\rangle}_{\langle k\rangle\langle\ell\rangle} = c^{\langle 2k-m\rangle}_{\langle k\rangle\langle\ell\rangle} \tag{4.10}$$

Exercises

Show that

$$\begin{aligned}\langle 4\rangle\circ\langle 6\rangle =& \langle 10\rangle+\langle 9\rangle+2\langle 8\rangle+2\langle 7\rangle+3\langle 6\rangle+2\langle 5\rangle\\ &+2\langle 4\rangle+\langle 3\rangle+\langle 2\rangle\end{aligned}$$

and hence for $S(12)$

$$\{84\}\circ\{6^2\} = \{10\ 2\}+\{84\}+\{6^2\}$$

Check that the above result is dimensionally correct.

Symmetric Functions and the Symmetric Group 5

B. G. Wybourne

To do research you don't have to know everything
All you have to know is one thing that is not known
–Art Schawlow *Nobel Laureate*

■ 5.1 $S-$function series

Infinite series of $S-$functions play an important role in determining branching rules and furthermore lead to concise symbolic methods well adapted to computer implementation. Consider the infinite series

$$L = \prod_{i=1}^{\infty}(1-x_i) = 1 - \sum x_1 + \sum x_1x_2 - \ldots \tag{5.1}$$

where the summations are over all distinct terms. e.g.

$$\sum x_1x_2 = x_1x_2 + x_1x_3 + \ldots + x_2x_3 + x_2x_4 + \ldots \tag{5.2}$$

Recalling the definition of elementary symmetric funvctions we see that Eq.(5.2) is simply a signed sum over an infinite set of elementary symmetric functions e_n with

$$e_n = m_{1^n} = s_{1^n} = \{1^n\} \tag{5.3}$$

and hence Eq.(5.2) may be written as an infinite sum of $S-$functions such that

$$L = 1 - \{1\} + \{1^2\} - \{1^3\} + \ldots = \sum_{m=0}^{\infty}(-1)^m\{1^m\} \tag{5.4}$$

We may define a further infinite series of $S-$functions by taking the inverse of Eq.(5.2) to get

$$M = \prod_{i=1}^{\infty}(1-x_i)^{-1} = 1 + \{1\} + \{2\} + \ldots = \sum_{m=0}^{\infty}\{m\} \tag{5.5}$$

Clearly

$$LM = 1 \tag{5.6}$$

a result that is by no means obvious by simply looking at the product of the two series.

In practice large numbers of infinite series and their associated generating functions may be constructed.

We list a few of them below:

$\mathrm{A} = \sum_\alpha (-1)^{w_\alpha}\{\alpha\}$	$\mathrm{B} = \sum_\beta \{\beta\}$
$\mathrm{C} = \sum_\gamma (-1)^{w_\gamma/2}\{\gamma\}$	$\mathrm{D} = \sum_\delta \{\delta\}$
$\mathrm{E} = \sum_\epsilon (-1)^{(w_\epsilon+r)/2}\{\epsilon\}$	$\mathrm{F} = \sum_\zeta \{\zeta\}$
$\mathrm{G} = \sum_\epsilon (-1)^{(w_\epsilon-r)/2}\{\epsilon\}$	$\mathrm{H} = \sum_\zeta (-1)^{w_\zeta}\{\zeta\}$
$\mathrm{L} = \sum_m (-1)^m\{1^m\}$	$\mathrm{M} = \sum_m \{m\}$
$\mathrm{P} = \sum_m (-1)^m\{m\}$	$\mathrm{Q} = \sum_m \{1^m\}$

(5.7)

where (α) and (γ) are mutually conjugate partitions, which in the Frobenius notation take the form

$$(\alpha) = \begin{pmatrix} a_1 & a_2 & \dots & a_r \\ a_1+1 & a_2+1 & \dots & a_r+1 \end{pmatrix} \tag{5.8a}$$

and

$$(\gamma) = \begin{pmatrix} a_1+1 & a_2+1 & \dots & a_r+1 \\ a_1 & a_2 & \dots & a_r \end{pmatrix} \tag{5.8b}$$

(δ) is a partition into *even parts* only and (β) is conjugate to (δ). (ζ) is any partition and (ϵ) is any self-conjugate partition. r is the Frobenius rank of (α), (γ) and (ϵ).
These series occur in mutually inverse pairs:

$$AB = CD = EF = GH = LM = PQ = \{0\} = 1 \tag{5.9}$$

Furthermore,

$$\begin{aligned} LA = PC = E \quad & MB = QD = F \\ MC = AQ = G \quad & LD = PB = H \end{aligned} \tag{5.10}$$

We also note the series

$$R = \{0\} - 2\sum_{a,b}(-1)^{a+b+1}\begin{pmatrix} a \\ b \end{pmatrix} \qquad S = \{0\} + 2\sum_{a,b}\begin{pmatrix} a \\ b \end{pmatrix} \tag{5.11}$$

where we have again used the Frobenius notation, and

$$\begin{aligned} V = \sum_\omega (-1)^q\{\tilde\omega\} \quad & W = \sum_\omega (-1)^q\{\omega\} \\ X = \sum_\omega \{\tilde\omega\} \quad & Y = \sum_\omega \{\omega\} \end{aligned} \tag{5.12}$$

where (ω) is a partition of an even number into at most two parts, the second of which is q, and $\tilde\omega$ is the conjugate of ω. We have the further relations

$$RS = VW = \{0\} = 1 \tag{5.13}$$

and

$$\begin{aligned} PM = AD = W \quad & LQ = BC = V \\ MQ = FG = S \quad & LP = HE = R \end{aligned} \tag{5.14}$$

■ 5.2 Symbolic manipulation

The above relations lead to a method of describing many of the properties of groups via symbolic manipulation of infinite series of $S-$functions. Thus if $\{\lambda\}$ is an $S-$function then we may symbolically write, for example,

$$\{\lambda/M\} = \sum_m \{\lambda/m\} \tag{5.15}$$

We can construct quite remarkable identities such as:

$$BD = \sum_\zeta \{\zeta\}\cdot\{\zeta\} \tag{5.16}$$

or for an arbitrary $S-$function $\{\epsilon\}$

$$BD \cdot \{\epsilon\} = \sum_{\zeta} \{\zeta\} \cdot \{\zeta/\epsilon\} \tag{5.17}$$

Equally remarkably we can find identities such as

$$\{\sigma \cdot \tau\}/Z = \{\sigma/Z\} \cdot \{\tau/Z\} \quad \text{for} \quad Z = L, M, P, Q, R, S, V, W \tag{5.18a}$$

$$\{\sigma \cdot \tau\}/Z = \sum_{\zeta} \{\sigma/\zeta Z\} \cdot \{\tau/\zeta Z\} \quad \text{for} \quad Z = B, D, F, H \tag{5.18b}$$

$$\{\sigma \cdot \tau\}/Z = \sum_{\zeta} (-1)^{w_\zeta} \{\sigma/\zeta Z\} \cdot \{\tau/\tilde{\zeta} Z\} \quad \text{for} \quad Z = A, C, E, G \tag{5.18c}$$

These various identities can lead to a symbolic method of treating properties of groups particularly amenable to computer implementation.

■ References

For more details see:

1. R C King, Luan Dehuai and B G Wybourne, *Symmetrized powers of rotation group representations* J Phys A: Math. Gen.**14** 2509 (1981)
2. G R E Black, R C King and B G Wybourne, *Kronecker products for compact semisimple Lie groups* J Phys A: Math. Gen. **16** 1555 (1983)
3. R C King, B G Wybourne and M Yang, *Slinkies and the $S-$function content of certain generating functions* J Phys A: Math. Gen. **22** 4519 (1989)

■ 5.3 The $U_n \to U_{n-1}$ branching rule

As an illustration of the preceding remarks we apply the properties of $S-$functions to the determination of the $U_n \to U_{n-1}$ branching rules. The vector irrep $\{1\}$ of U_n can be taken as decomposing under $U_n \to U_{n-1}$ as

$$\{1\} \to \{1\} + \{0\} \tag{5.19}$$

that is into a vector $\{1\}$ and scalar $\{0\}$ of U_{n-1}. In general, the spaces corresponding to tensors for which a particular number of indices, say m, take on the value n, define invariant subspaces. Such indices must be mutually symmetrised. The irreducible representations specified by the quotient $\{\lambda/m\}$ are those corresponding to tensors obtained by contracting the indices of the tensor corresponding to $\{\lambda\}$ with an $m-$th rank symmetric tensor. Thus we may symbolically write the general branching rule as simply

$$\{\lambda\} \to \{\lambda/M\} \tag{5.20}$$

Thus for example under $U_3 \to U_2$ we have

$$\begin{aligned} \{21\} &\to \{21/M\} \\ &\to \{21/0\} + \{21/1\} + \{21/2\} \\ &\to \{21\} + \{2\} + \{11\} + \{1\} \end{aligned} \tag{5.21}$$

■ 5.4 The Gel'fand states and the betweenness condition

The so-called Gel'fand states play an important role in the Unitary Group Approach (UGA) to many-electron theory. This comes about from considering the canonical chain of groups

$$U_n \supset U_{n-1} \supset \ldots U_2 \supset U_1 \tag{5.22}$$

The states of such a chain follow directly from consideration of Eq.(5.20). Each state may be represented by a triangular array having n rows. There are n entries $m_{i,n}$ with $i = 1, 2, \ldots, n$ corresponding to the usual partition (λ) padded out with zeroes to fill the row if need be. The second row contains $n-1$ entries $m_{i,n-1}$ placed below the first row so that the entry $m_{1,n-1}$ occurs between the entries $m_{1,n}$ and $m_{2,n}$ etc. Each successive row contains one less entry with the bottom row containing just one entry $m_{1,1}$. The number of such states is just the dimension of the irrep $\{\lambda\}$ of U_n.

Consider the irrep of U_3 labelled as $\{21\}$. We find the eight Gel'fand states

$$\begin{pmatrix} 2 & & 1 & & 0 \\ & 2 & & 1 & \\ & & 2 & & \end{pmatrix} \quad \begin{pmatrix} 2 & & 1 & & 0 \\ & 2 & & 1 & \\ & & 1 & & \end{pmatrix}$$

$$\begin{pmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 2 & & \end{pmatrix} \quad \begin{pmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{pmatrix}$$

$$\begin{pmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 0 & & \end{pmatrix} \quad \begin{pmatrix} 2 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{pmatrix}$$

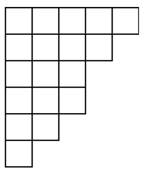
$$\begin{pmatrix} 2 & & 1 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{pmatrix} \quad \begin{pmatrix} 2 & & 1 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{pmatrix}$$

■ 5.5 The Murnaghan-Nakayama rule for $S(N)$ characters

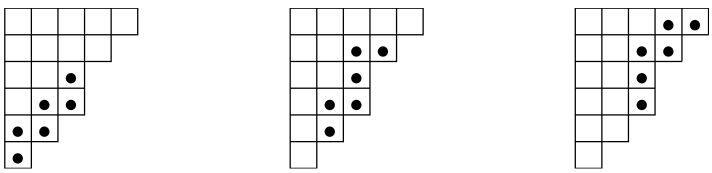
It is not my intention to give anymore than hints at methods of calculating the characters of $S(N)$ a subject well covered in the books of James and Kerber, Littlewood, Murnaghan, Macdonald, Robinson and Sagan but rather to indicate those specialisations that are of immediate application in quantum chemistry. The Murnaghan-Nakayama rule is of particular value in starting practical calculations. The key concept is that of the removal of *rim hook*s or *continuous boundary strip*s from a Young frame. A rim hook is a continuous strip of cells along the boundary of the Young frame which when removed leaves a standard Young frame. The length of the strip is the total number of cells in the rim hook. We associate a *sign* with a given rim hook. If the rim hook involves v cells in the vertical direction then the sign of the rim hook is

$$sgn = (-1)^{v-1} \tag{5.23}$$

As an example consider the Young frame associated with the partition (543321)



Let us now mark the three permissible continuous boundary hooks of length 6 as below



In each case the 6-hook involves four rows and hence the number of vertical cells is $v = 4$ and hence the sign is $sgn = -1$.

■ *The Murnaghan-Nakayama Algorithm* The characteristic $\chi^{\{\lambda\}}_{(\rho)}$ for $S(N)$, where $\{\lambda\}$ is the irrep and (ρ) the class may be determined by

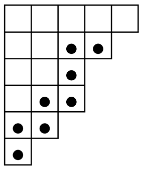
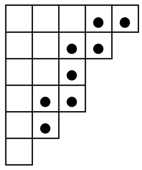
1. Draw the Young frame for the partition λ.
2. Set $i = 1$. Set $sgn\ = +1$.
3. While $\rho_i <> 0$ do begin
4. Remove a rim hook of length ρ_i in all possible ways that leave a standard Young frame. If this is not possible for any of the Young frames then $\chi^{\{\lambda\}}_{(\rho)} = 0$ and the algorithm is terminated.
5. A sign $sgn = sgn * newsign$ is to be associated with each new Young frame created in 3. with $newsign$ being the sign of the rim hook being removed.

6. Set $i = i + 1$
7. End
8. The characteristic $\chi^{\{\lambda\}}_{(\rho)}$ is equal to the sum of the signed units at the termination of the loop.

NB. The result is independent of the order of the removal of the rim hooks.

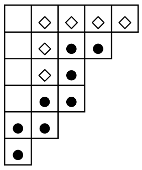
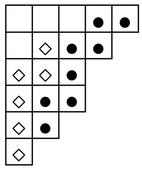
Example of $\chi^{\{543321\}}_{(864)}$

First remove a rim hook of length 8 from the Young frame as shown below

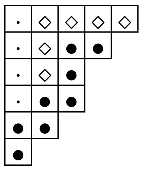
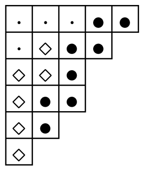



In each case the sign of the 8−hook is positive.

Now remove the 6−hook from each of the above two frames to give

Again each 6−hook has a positive sign. Now remove a 4−hook from each frame to give

The sign of each 6−hook is negative and hence each of the frames yields an overall negative sign and hence

$$\chi^{\{543321\}}_{(864)} = -2$$

■ 5.6 The characteristics $\chi^{\{\lambda\}}_{(N)}$

The characteristics $\chi^{\{\lambda\}}_{(N)}$ constitute an important special case. By the Murnaghan-Nakayama rule there is just a single rim-hook of length N to be removed. The only possibility for a non-zero characteristic is if the frame of the partition λ is a single hook of the form $(a1^b)$ with $N = a + b$. The characteristic is thus either null or ± 1. Precisely

$$\chi^{\{\lambda\}}_{(N)} = \begin{cases} (-1)^b & \text{if } \lambda = (a+1, 1^b) \\ 0 & \text{otherwise} \end{cases} \tag{5.24}$$

■ 5.7 The power sum symmetric functions and $S(N)$ characters

The character table of $S(N)$ is the transition matrix $M(p, s)$ that expresses power sum symmetric functions p_ρ as a linear combination of $S-$functions s_λ with $|\rho| = |\lambda| = N$. Thus

$$p_\rho = \sum_\lambda \chi^\lambda_\rho s_\lambda \tag{5.25}$$

We have the important special case

$$p_n = \sum_{\substack{a,b=0 \\ a+b+1=n}}^{n-1} (-1)^b s_{a+1,1^b} \tag{5.26}$$

Recalling that the power sum symmetric functions are multiplicative we can use Eq. (5.26) to compute all the characteristics associated with a given class by simple application of the Littlewood-Richardson rule. As an example consider the characteristics for the class (31) of $S(4)$. From Eq. (5.26) we have

$$\begin{aligned} p_3 &= \{3\} - \{21\} + \{1^3\} \\ p_1 &= \{1\} \end{aligned}$$

and hence

$$\begin{aligned} p_{31} &= (\{3\} - \{21\} + \{1^3\}) \cdot (\{1\}) \\ &= \{4\} - \{2^2\} + \{1^4\} \end{aligned}$$

showing immediately that the only non-zero characteristics associated with the class (31) are

$$\chi_{31}^{4} = +1, \qquad \chi_{31}^{2^2} = -1, \qquad \chi_{31}^{1^4} = +1$$

Exercises

1. Generalize the power sum symmetric function to

$$p_n(q;t) = \sum_{\substack{a,b=0 \\ a+b+1=n}}^{n-1} (-1)^q q^a s_{a+1,1^b}(x) \tag{27}$$

and show that

$$p_{31}(q;x) = q^2\{4\} + (q^2-1)\{31\} - q\{2^2\} - (q-1)\{21^2\} + \{1^4\}$$

and for $q = 1$ the $S(4)$ result is recovered. This takes one into Hecke algebras. ([KW1]King and Wybourne, *J. Phys. A: Math. Gen.* **23**, L1193(1990); [KW2]*J. Math, Phys.* **33**, 4 (1992).).

2. Construct a $q-$dependent character table for $N = 3$ and compare it with the corresponding table for $S(3)$. See [KW1].

"It did, Mr Widdershins, until quantum mechanics came along. Now everything's atoms. Reality is a fuzzy business, Mr Widdershins. I see with my eyes, which are a collection of whirling atoms, through the light, which is a collection of whirling atoms. What do I see? I see you Mr Widdershins, who are also a collection of whirling atoms. And in all this intermingling of atoms who is to know where anything starts and anything stops. It's an atomic soup we're in, Mr Widdershins. And all these quantum limbo states only collapse into one concrete reality when there is a human observer"

Pauline Melville, *The Girl with the Celestial Limb* (1991)

Symmetric Functions and the Symmetric Group 6
B. G. Wybourne

This is why I value that little phrase "I don't know" so highly. It's small, but it flies on mighty wings. It expands our lives to include the spaces within us as well as those outer expanses in which our tiny Earth hangs suspended. If Isaac Newton had never said to himself "I don't know," the apples in his little orchard might have dropped to the ground like hailstones and at best he would have stooped to pick them up and gobble them with gusto. Had my compatriot Marie Sklodowska-Curie never said to herself "I don't know", she probably would have wound up teaching chemistry at some private high school for young ladies from good families, and would have ended her days performing this otherwise perfectly respectable job. But she kept on saying "I don't know," and these words led her, not just once but twice, to Stockholm, where restless, questing spirits are occasionally rewarded with the Nobel Prize.

WISLAWA SZYMBORSKA (Nobel Lecture 1996)

■ 6.1 Plethysm of $S-$functions

The plethysm of $S-$functions is a property that has many important applications in symmetry aspects of many-body problems in physics and grew out of the mathematical theory of invariants though nowadays forms an integral part of combinatorial mathematics. There is a close connection between the plethysm of $S-$functions and branching rules.

Let $\Lambda^n = \Lambda^n(x_1, \ldots, x_N)$ denote the space of homogeneous symmetric polynomials of degree n.Then given symmetric polynomials with integer coefficients

$$\mathrm{P} \in \Lambda^n \qquad \text{and} \qquad \mathrm{Q} \in \Lambda^m$$

then

$$\mathrm{P[Q]} \qquad \text{is a symmetric polynomial in} \qquad \Lambda^{mn} \tag{6.1}$$

In this sense a plethysm can be seen as a substitution process. As a simple example consider the power sum symmetric functions

$$p_n = \sum_i x_i^n \qquad \text{and} \qquad p_m = \sum_i x_i^m$$

then

$$p_n[p_m] = p_m[p_n] = p_{mn} \tag{6.2}$$

Likewise

$$p_n[e_m] = e_m[p_n] = m_n^m \tag{6.3}$$

and

$$p_n[m_\mu] = m_\mu[p_n] = m_{\mu\cdot n} \tag{6.4}$$

where $\mu \cdot n$ signifies that each part of μ is multiplied by the integer n.

The above examples are all commutative which is not the general case. In general the $S-$function content of $s_\lambda[s_\mu]$ is not the same as that of $s_\mu[s_\lambda]$.

As an example of $S-$function plethysm consider the evaluation of $s_2[s_{1^2}](x_1, \ldots, x_4)$. We express $s_{1^2}(x_1, \ldots, x_4)$ as a sum of monomials,

$$s_{1^2}(x_1, \ldots, x_4) = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 \tag{6.5}$$

Now regard s_2 as a function in as many monomials as in (6.5) i.e.

$$s_2[s_{1^2}](x_1, \ldots, x_4) = s_2(x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4) \tag{6.6}$$

Very tediously, the right-hand-side of (6.6) may be expanded as a sum of monomials which in turn may be expressed in terms of $S-$functions to yield, finally

$$s_2[s_{1^2}](\mathbf{x}) = s_{2^2}(\mathbf{x}) + s_4(\mathbf{x}) \tag{6.7}$$

Noting that

$$s_2(x_1, \ldots, x_4) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 \tag{6.8}$$

We have

$$s_{1^2}[s_2](x_1,\ldots,x_4) = s_{1^2}(x_1^2+x_2^2+x_3^2+x_4^2+x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4) \quad (6.9)$$

which may be expanded as a sum of monomials and then into $S-$functions to give

$$s_{1^2}[s_2](\mathbf{x}) = s_{31}(\mathbf{x}) \quad (6.10)$$

which is different from (6.7).

While the above examples have involved just four variables the results actually hold for any number of variables $n \geq 4$.

■ Exercise

1. Show that $s_{1^2}[s_{1^2}](\mathbf{x}) = s_{21^2}(\mathbf{x})$

■ 6.2 Plethysm Notation

The plethysm of $S-$functions was introduced by D E Littlewood in terms of invariant matrices and who used the notation $\{\lambda\}\otimes\{\mu\}$. This notation is used almost universally by physicists whereas the corresponding plethysm viewed as an $S-$function substitution is almost universally written by combinatorists as $s_\mu[s_\lambda]$ the corrspondence between the two notations being

$$\{\lambda\}\otimes\{\mu\} \equiv s_\mu[s_\lambda] \quad (6.11)$$

It much that follows we shall use the physicists notation.

■ 6.3 The algebra of plethysms

The algebra of plethysms is governed by the rules

$$A\otimes(B\pm C) = A\otimes B \pm A\otimes C \quad (6.12a)$$

$$A\otimes(BC) = (A\otimes B)(A\otimes C) \quad (6.12b)$$

$$A\otimes(B\otimes C) = (A\otimes B)\otimes C \quad (6.12c)$$

$$(A+B)\otimes\{\mu\} = \sum_\zeta (A\otimes\{\mu/\zeta\})(B\otimes\{\zeta\}) \quad (6.12d)$$

$$(A-B)\otimes\{\mu\} = \sum_\zeta (-1)^{w_\zeta}(A\otimes\{mu/\zeta\})(B\otimes\{\tilde{\zeta}\}) \quad (6.12e)$$

$$(AB)\otimes\{\mu\} = \sum_\rho (A\otimes\{\rho\})(B\otimes\{\mu\circ\rho\}) \quad (6.12f)$$

Note that (6.12c) shows the associativity of the plethysm operation and that in (6.12f) the $\circ$ signifies an inner product of $S-$functions so that $\{\mu\}\circ\{\rho\}$ is the Kronecker product of irreducible representations of S_m labelled μ and ρ which are both partitions of m. In (6.12e) w_ζ is the weight of the partition (ζ) and the partition $(\tilde{\zeta})$ is conjugate to (ζ).

■ 6.4 Plethysm and $S-$function series

Later we shall show that plethysm gives a powerful tool for developing symbolic representations of branching rules for going from the representation of a group $\mathcal{G}$ to those of a subgroup $\mathcal{H}$. However, we must first consider plethysm and $S-$function series. The basic ideas are developed in

1. M Yang and B G Wybourne, *New $S-$function series and non-compact Lie groups* J Phys A:Math.Gen. **19** 3513 (1986)
2. R C King, B G Wybourne and M Yang, *Slinkies and the $S-$function content of certain generating functions* J Phys A:Math.Gen. **22** 4519 (1989)

Consider the infinite $S-$function series

$$L(x) = \prod_{i=1}^{\infty}(1-x_i) \quad (6.13a)$$

$$= \sum_{m=0}^{\infty}(-1)^m\{1^m\} \quad (6.13b)$$

The inverse L^{-1} is

$$L^{-1} = \left(\prod_{i=1}^{\infty}(1-x_i)\right)^{-1} = \prod_{m=0}^{\infty}\{m\} = M \tag{6.14}$$

Let us define the adjoint series $L^{\dagger}$ as the conjugate ($\tilde{}$) inverse or the inverse conjugate of L:

$$L^{\dagger} = (\tilde{L})^{-1} = \tilde{L}^{-1} \tag{6.15}$$

leading to

$$L^{\dagger} = \prod_{i=1}^{\infty}(1+x_i) = \sum_{m=0}^{\infty}\{1^m\} = Q \tag{6.16}$$

Note that taking the adjoint ($\dagger$) is equivalent to the substitution

$$x_i \to -x_i \tag{6.17}$$

in $L(x_i)$, which can be viewed as a plethysm:

$$L^{\dagger} = L(-x_i) = (-\{1\}) \otimes L \tag{6.18}$$

The conjugate of L is also the inverse of $L^{\dagger}$ and hence

$$\tilde{L} = (L^{\dagger})^{-1} = \left(\prod_{i=1}^{\infty}(1+x_i)\right)^{-1} = \prod_{m=0}^{\infty}(-1)^m\{m\} = P \tag{6.19}$$

We thus have four infinite $S-$function series L, M, P, Q related by the four properties, identity (I), conjugation ($\tilde{}$), inverse (-1) and adjoint ($\dagger$) which form a discrete four-element group with the Cayley table

	I	$\tilde{}$	-1	$\dagger$
I	I	$\tilde{}$	-1	$\dagger$
$\tilde{}$	$\tilde{}$	I	$\dagger$	-1
-1	-1	$\dagger$	I	$\tilde{}$
$\dagger$	$\dagger$	-1	$\tilde{}$	I

Having obtained the four L type series we can obtain further series by simple substitution into the L series. Thus under the substitution

$$x_i \to x_i x_j \quad (i<j) \tag{6.20}$$

we obtain

$$L(x_i x_j) = \prod_{(i<j)}^{\infty}(1-x_i x_j) \tag{6.21a}$$

$$= \{1^2\} \otimes L \tag{6.21b}$$

$$= \sum_{\alpha}(-1)^{w_\alpha}\{\alpha\} = A \tag{6.21c}$$

where in the Frobenius notation

$$(\alpha) = \begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_r \\ \alpha_1+1 & \alpha_2+1 & \dots & \alpha_r+1 \end{pmatrix} \tag{6.22}$$

Continuing we could construct four series $A,\ A^{-1},\ \tilde{A},\ A^{\dagger}$.

The substitution

$$x_i \to x_i^2 \tag{6.23}$$

leads to

$$L(x_i^2) = \prod_{i=1}^{\infty}(1 - x_i^2) \tag{6.24a}$$

$$= (\{2\} - \{1^2\}) \otimes L \tag{6.24b}$$

$$= \sum_{p,q=0}^{\infty} (-1)^p \{p + 2q, p\} = V \tag{6.24c}$$

■ 6.5 Why are infinite $S-$function series important?

We noted earlier that $S-$functions can be related to the characters of the unitary groups $U(n)$ and the $S-$function multiplication via the Littlewood-Richardson rule corresponds to the resolution of Kronecker products of irreducible representations in $U(n)$. We also noted that a given irreducible representation of $U(n)$, say $\{\lambda\}$ becomes reducible under the group-subgroup restriction $U(n) \to U(n-1)$ such that

$$\{\lambda\} \to \{\lambda/M\} \tag{6.25}$$

where M is the infinite $S-$function series

$$M = \sum_{m=0}^{\infty} \{m\} \tag{6.26}$$

The number of terms is rendered finite by the occurrence of the M series as a $S-$function skew. The irreducible representations of $U(n)$ are all finite dimensional so the occurrence of the $S-$function series as skews is to be expected. However, there are, so-called non-compact groups whose non-trivial unitary representations are infinite-dimensional. In those cases the characters may be represented in terms of infinite $S-$function series and upon restriction to compact subgroups the branching rules involving an infinite number of representations of the compact subgroup and the $S-$function series appear in the numerator rather than as skews. Likewise whereas for compact groups, like $U(n)$, the Kronecker products involve a finite number of terms for the noncompact groups a Kronecker product of a pair of infinite-dimensional irreducible representations will usually involve an infinite number of infinite dimensional unitary irreducible representations. We shall not explore non-compact groups in any detail here. The interested reader may explore some of the references below.

1. D. J. Rowe, B. G. Wybourne and P. H. Butler, *Unitary Representations, Branching Rules and Matrix Elements for the Non-Compact Symplectic Groups,* J Phys A:Math.Gen. **18**, 939-953 (1985)
2. R. C. King and B. G. Wybourne, *Holomorphic Discrete Series and Harmonic Series Unitary Irreducible Representations of Non-Compact Lie Groups:* $Sp(2n,\Re)$, $U(p,q)$ *and* $SO^*(2n)$, J Phys A:Math.Gen. **18**, 3113-3139 (1985)
3. B. G. Wybourne, *The representation space of the nuclear symplectic* $Sp(6,\Re)$ *shell model* J Phys A:Math.Gen. **25**, 4389-4398 (1992)
4. K. Grudzinski and B. G. Wybourne, *Plethysm for the noncompact group* $Sp(2n,\Re)$ *and new* $S-$*function identities* J Phys A:Math.Gen. **29**, 6631-6641 (1996).
5. Jean-Yves Thibon, Frederic Toumazet and Brian G Wybourne, *Products and plethysms for the fundamental harmonic series representations of* $U(p,q)$, J Phys A:Math.Gen. **30**, 4851-6 (1997)
6. Jean-Yves Thibon, Frederic Toumazet and Brian G Wybourne, *Symmetrised squares and cubes of the fundamental unirreps of* $Sp(2n,\Re)$, J Phys A:Math.Gen. **31**, 1073-86 (1998)
7. R C King and B G Wybourne, Products and symmetrised powers of irreducible representations of $Sp(2n,\Re)$ and their associates, **31**,6669-6689 (1998)
8. R C King, F. Toumazet and B G Wybourne, *Products and symmetrised powers of irreducible representations of* $SO^*(2n)$, J Phys A:Math.Gen. **31**, 6691-6705 (1998)
9. R C King and B G Wybourne, *Analogies between finite-dimensional irreps of* $SO(2n)$ *and infinite-dimensional irreps of* $Sp(2n,\Re)$ *Part I: Characters and products* , J.Math.Phys. **41**, 5002-19 (2000)

10. R C King and B G Wybourne, *Analogies between finite-dimensional irreps of $SO(2n)$ and infinite-dimensional irreps of $Sp(2n, \Re)$ Part II:Plethysms*, J.Math.Phys. **41**,5656-90 (2000)

■ 6.7 Regular matrix groups

Consider square $n \times n$ matrices A such that

1. The unit element is the $n \times n$ identity matrix,

$$\mathrm{I} = \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & 1 \end{pmatrix} \tag{27}$$

2. The existence of an inverse element, A^{-1}, is assured by restriction to non-singular matrices such that

$$\det |A| \neq 0 \tag{6.28}$$

3. The laws of matrix multiplication are such that the associative law of multiplication is satisfied.
4. The set of matrices is such that closure is assured.

If the above four properties are satisfied then the set of matrices will form a group. Groups involving regular matrices may be finite or infinite, be discrete or continuous, and have real ($\Re$) or complex (C) elements. The variables in the real space $\Re^n$ will be designated $\mathbf{x} \equiv (x_1, \ldots, x_n)$ and in the complex space C^n as $\mathbf{z} \equiv (z_1, \ldots, z_n)$. A regular matrix of degree n acting in ($\Re^n$) or in C^n will produce transformations $\mathbf{x} \to \mathbf{x}'$ or $\mathbf{z} \to \mathbf{z}'$. In problems in physics we are frewquently interested classes of transformations that leave invariant some functional form of $\mathbf{x}$ or $\mathbf{z}$.

■ 6.8 Continuous matrix groups

Consider a group whose elements comprise all regular nonsingular real matrices of degree 2,

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{6.29}$$

Apart from the constraint

$$a_{11}a_{22} \neq a_{12}a_{21} \tag{6.30}$$

that follows from (6.28), the range of the elements of the matrix is unrestricted and we can parameterise the matrix elements a_{ij} as

$$a_{ij} = \delta_{ij} + \alpha_{ij} \tag{6.31}$$

If all $\alpha_{ij} = 0$ we simply obtain the identity matrix

$$\mathrm{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{6.32}$$

We can treat the α_{ij} as real independent parameters and generate all the elements of the group by a continuous variation of the α_{ij}. The range of the parameters is unbounded and limited only to the extent demanded by (6.30). Any element of the group can be designated by giving its associated values of the parameters α_{ij}.

■ Exercises

1. Show that the transformations produced by the matrices

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \qquad (0 \leq \theta < 2\pi) \tag{6.33}$$

acting in $\Re^2$ leave invariant the form $x_1^2 + x_2^2$.

2. Show that the transformations produced by the matrices

$$\begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \tag{6.34}$$

leave invariant the real quadratic form $x_1^2 - x_2^2$.

■ 6.9 Matrix groups - Examples

The general linear groups $GL(n, C)$ and $GL(n, \Re)$

The *complex general linear group* $GL(n, \mathrm{C})$ is the group of regular invertible complex matrices of degree n. A particular matrix is characterised by its n^2 elements with each element containg a real and an imaginary part. The continuous variation of the $2n^2$ parts (i.e. n^2 real and n^2 complex parts) will generate the entire group and hence the group is of dimension $2n^2$ and may be characterised by $2n^2$ real parameters.

If the elements of $GL(n, \mathrm{C})$ are restricted to real values only, then

$$GL(n, \mathrm{C}) \supset GL(n, \Re) \tag{6.35}$$

The special linear groups $SL(n, C)$ and $SL(n, \Re)$

These groups occur as subgroups of $GL(n, \mathrm{C})$ and $GL(n, \Re)$ respectively when the requirement that the determinant of their matrices be of determinant $+1$. Clearly, $SL(n, \mathrm{C})$ becomes a $2(n^2 - 1)$ parameter group and $SL(n, \Re)$ a $(n^2 - 1)$ parameter group and

$$GL(n, \mathrm{C}) \supset SL(n, \mathrm{C}) \supset SL(n, \Re) \tag{6.36}$$

The special linear groups are often referred to as *special unimodular groups.*

The unitary groups

The unitary matrices A of degree n form the elements of the n^2-parameter *unitary group* $U(n)$ that leaves invariant the Hermitian form

$$\sum i = 1^n z_i z_j^* \tag{6.37}$$

Since the unitarity of the matrices A requires that

$$A^\dagger A = \mathrm{I} \tag{6.38}$$

the range of matrix elements a_{ij} is restricted by the requirement that

$$\sum_t a_{it} a_{tj}^* = \delta_{ij} \tag{6.39}$$

and hence $|a_{ij}|^2 \leq 1$. In this case the parameter domain is bounded and $U(n)$ is an example of a *compact* group.

The special unitary group $SU(n)$

If we limit our attention to unitary matrices of determinant $+1$ we obtain the (n^2-1)-parameter special unitary group $SU(n)$.

The orthogonal groups

The group of *complex* orthogonal matrices of degree n form a $n(n-1)$- parameter group $O(n, \mathrm{C})$. Since ${}^tAA = \mathrm{I}$ we have $|A| = \pm 1$ and thus the group decomposes into two disconnected pieces and we cannot pass continuously from one piece to the other. The orthogonal matrices of determinant $+1$ form a subgroup of $O(n, \mathrm{C})$, the $n(n-1)$-parameter *special complex orthogonal group* $SO(n, \mathrm{C})$ whose matrices leave invariant the complex quadratic form

$$\sum_{i=1}^n z_i^2 \tag{6.40}$$

The special real orthogonal groups $O(n, \Re)$ and $SO(n, \Re)$

The set of real orthogonal matrices of degree n forms the $n(n-1)/2$- parameter real orthogonal group $O(n, \Re)$ while the set of real orthogonal matrices of determinant $+1$ form the real special orthogonal group $SO(n, \Re)$. Again $O(n, \Re)$ contains two disconnected pieces with $SO(n, \Re)$ occurring as a subgroup. The real special orthogonal group holds invariant the real quadratic form

$$\sum_{i=1}^n x_i^2 \tag{6.41}$$

The Symplectic groups $Sp(n, C)$ and $Sp(n, \Re)$

The symplectic group $Sp(n, \mathrm{C})$ is the $2n(2n+1)$-parameter group of regular complex matrices which hold invariant the non-degenerate skew-symmetric bilinear form

$$\sum_{i=1}^{n}(x_i y_i' - x_i' y_i) \tag{6.42}$$

of two vectors $\mathbf{x} \equiv (x_1, \ldots, x_n, x_i', \ldots, x_n')$ and $\mathbf{y} \equiv (y_1, \ldots, y_n, y_i', \ldots, y_n')$. $GL(n, \mathrm{C}) \supset Sp(2n, \mathrm{C})$ and the matrices need not be unitary. Restriction to real matrices gives the $n(2n+1)$-parameter group $Sp(2n, \Re)$.

The symplectic group $sp(2n) = U(2n) \cup Sp(2n, C)$ is known as the *unitary symplectic group*. This group, like $Sp(2n, \Re)$, is a $n(2n+1)$ -parameter group. The symplectic groups occur only in even-dimensional spaces and find applications in many areas of physics.

Symmetric Functions and the Symmetric Group 7
B. G. Wybourne

And yet the mystery of mysteries is to view machines making machines; a spectacle that fills the mind with curious, and even awful, speculation.
Benjamin Disraeli: Coningsby (1844)

■ 7.1 Group-subgroup decompositions

NB. Herein we follow closely R C King, *Branching rules for classical Lie groups using tensor and spinor methods*, J Phys A:Math.Gen. **8**,429 (1975). Branching rules play an important role in applications of group theory to problems in physics.Consider a group G with elements $\{g,\ldots\}$ and irreducible representations $\{\lambda_{\rm G},\ldots\}$ and a subgroup H i.e. G $\supset$ H with elements $\{h,\ldots\}$ and irreducible representations $\{\mu_{\rm H},\ldots\}$. The restriction of the set of matrices $\{\lambda_{\rm G}(g)\}$ forming the representation $(\lambda)_{\rm G}$ of G to the set$\{\lambda_{\rm G}(h)\}$ yields a representation of H which is generally reducible. If

$$\lambda_{\rm G}(h)=\sum_{\mu_{\rm H}} m_{\lambda_{\rm G}}^{\mu_{\rm H}}\mu_{\rm H}(h) \quad \text{for all} \quad h\in {\rm H} \tag{7.1}$$

then under G $\downarrow$ H we have the *branching rule*

$$ {\rm G}\downarrow{\rm H} \qquad (\lambda)_{\rm G}\downarrow\sum_{\mu_{\rm H}} m_{\lambda_{\rm G}}^{\mu_{\rm H}}(\mu)_{\rm H} \tag{7.2}$$

where the $m_{\lambda_{\rm G}}^{\mu_{\rm H}}$ are known as the *branching rule multiplicities.*

■ 7.2 The Unitary,$U(n)$, and Special Unitary, $SU(n)$, groups

We have already noted the relationship between $S-$functions and the characters of the unitary group and the fact that the irreducible representations of $U(n)$ may be labelled by ordered partitions of integers, thus $\{\lambda\}\equiv\{\lambda_1,\ldots,\lambda_n\}$. (NB in addition to these *covariant* irreducible representations there are irreducible representations involving both covariant and contravariant indices - see the above reference for more details). In practice zero parts are omitted. Under the restriction $U(n)\downarrow SU(n)$ an irreducible representation $\{\lambda\}$ of $U(n)$ remains irreducible and hence we may still label the irreducible representations of $SU(n)$ by ordered partitions with the proviso that irreducible representations of $U(n)$ involving partitions with n positive integers are equivalent to an irreducible representation of $SU(n)$ involving fewer than n positive integers. The equivalence is such that

$$\{\lambda_1,\ldots,\lambda_n\}\equiv\{\lambda_1-\lambda_n,\ldots,0\} \quad if\ \lambda_n>0 \tag{7.3}$$

Thus for $SU(n)$ the irreducible representations involve at most $n-1$ positive integers. Thus under $U(3)\downarrow SU(3)$ we have, for example, the equivalences

$$\{321\}\equiv\{21\},\quad \{432\}\equiv\{21\},\quad \{1^3\}\equiv\{0\}$$

NB. irreducible representations that are inequivalent in $U(n)$ may, under the restriction $U(n)\downarrow SU(n)$ be equivalent to the *same* $SU(n)$ irreducible representation. Irreducible representations of $SU(n)$ that involve partitions $\{\lambda\}$ such that

$$\{\lambda_1,\lambda_2,\ldots,\lambda_n\}\equiv\{\lambda_1-\lambda_n,\lambda_2-\lambda_{n-1},\ldots,0\} \tag{7.4}$$

are said to be *contragredient* to one another and are of the same dimension. Thus in $SU(3)$ we have the pairs $\{2\},\{2^2\}$, $\{31\},\{32\}$ etc. Such pairs are sometimes labelled as $\{\lambda\},\{\bar\lambda\}$. If $\{\lambda\}\equiv\{\bar\lambda\}$ then $\{\lambda\}$ is said to be *self-contragredient*. Thus in $SU(3)$ $\{21\}$, $\{42\}$ are examples of self-contragredient irreducible representations.

■ 7.3 Kronecker products in $U(n)$ and $SU(n)$

The Kronecker product of a pair of irreducible representations of $U(n)$, say $\{\lambda\}$ and $\{\mu\}$ may be resolved into a sum of irreducible representations of $U(n)$ by use of the Littlewood-Richardson rule for $S-$functions to give

$$\{\lambda\} \times \{\mu\} = \sum_{\nu} c^{\nu}_{\lambda\mu}\{\nu\} \tag{7.5}$$

with the proviso that all $\{nu\}$ involving more than n non-zero parts are to be discarded. For example, in $U(3)$ we have upon application of the Littlewood-Richardson rule

$\{21\} \times \{31\} =$

$$\{321^2\} + \{32^2\} + \{3^21\} + \{41^3\} + 2\{421\} + \{43\} + \{51^2\} + \{52\} \tag{7.6}$$

However, the four part partitions $\{321^2\} + \{41^3\}$ must be discarded as null in $U(3)$. NB. for $n \geq 4$ holds as it stands. For the group $SU(3)$ (7.3) must be applied to the three part partitions occurring in (7.6) to give for $SU(3)$ $\{21\} \times \{31\} =$

$$\{1\} + \{2^2\} + 2\{31\} + \{4\} + \{43\} + \{52\} \tag{7.7}$$

■ Exercises

1. Verify that the product of the dimensions on the lhs of (7.7) is equal to the sum of the dimensions on the rhs.
2. Show that in $SU(3)$
$$\{1\} \times \{\bar{1}\} = \{0\} + \{21\}$$
3. Show that in $SU(3)$
$$\{1\} \times \{1\} \times \{1\} = \{0\} + 2\{21\} + \{3\}$$

■ 7.4 The labelling of irreducible representations for the classical Lie groups

Hereon we follow

1. G R E Black, R C King and B G Wybourne, *Kronecker products for compact semisimple Lie groups*, J Phys A:Math.Gen. **16**, 1555 (1983). (BKW)

The partition (λ) of weight w_λ serves to label an irreducible representation $\{\lambda\}$ of $U(n)$ and to specify the symmetry properties of the corresponding w_λth-rank covariant tensor forming a basis for the irreducible representation. This same covariant tensor forms a basis for representations of the subgroups of $U(n)$, including the orthogonal group, $O(n)$, and if n is *even*, the symplectic group $Sp(n)$. In general, these representations are reducible and will be labelled by the partitions $[\lambda]$ and $\langle\lambda\rangle$ respectively.

As well as the tensor irreducible representations labelled by $[\lambda]$, $O(n)$ also has double-valued or spinor representations denoted by $[\Delta; \lambda]$ where Δ is the fundamental spin representation of dimension $2^{[n/2]}$.

For all linear groups there exists amongst the irreducible representations a one-dimensional irreducible representation, denoted by ε, which maps each group element to the value of its determinant. By definition all the elements of $SU(n)$, $SO(n)$ and $Sp(n)$ have determinant $+1$, so that the irreducible representations ε coincide with the identity irreducible representations $\{0\}$, $[0]$ and $\langle 0\rangle$ respectively. However, for $U(n)$ and $O(n)$ this is not the case. For $U(n)$ ε is the irreducible representation $\{1^n\}$ with an inverse

$$\varepsilon^{-1} = \bar{\varepsilon} = \{\bar{1^n}\} \qquad (for\ U(n)) \tag{7.8}$$

For $O(n)$, all the group elements are of determinant ± 1 and hence

$$\varepsilon^{-1} = \varepsilon \qquad and \qquad \varepsilon \times \varepsilon \quad (for\ O(n)) \tag{7.9}$$

The product of ε with any irreducible representation is also an irreducible representation, and inequivalent irreps related by some power of ε are said to be *associated.* For $U(n)$ there are an infinite number of inequivalent associated irreducible representations associated with a given irreducible representation, one of which will be specified by a partition into less than n parts. For instance ... $\overline{\{6^2521\}}$, $\overline{\{5^241\}}$, $\overline{\{4^23;1\}}\ldots\{541\}$, $\{6521^2\}$ are all associated irreducible representations of $U(5)$.

Since under $U(n) \downarrow SU(n)$ $\varepsilon \downarrow \{0\}$ it follows that all mutually associated irreducible representations of $U(n)$ give equivalent irreducible representations of $SU(n)$.

In the case of $O(n)$ any given irreducible representation can possess at most one inequivalent associated irreducible representation. Irreducible representations for which the character is zero for all group elements of determinant -1 possess an associate that is equivalent to itself. Such irreducible representations are said to be *self-associate.* For $O(2k)$ all the spinor irreducible representations and all the tensor irreducible representations labelled by exactly k parts are self-associate. The remaining tensor irreducible representations of $O(2k)$ and all irreducible representations, tensor and spinor, of $O(2k+1)$ are not self-associate. Associated pairs of irreducible representations are denoted by $[\lambda]$ and $[\lambda]^*$ and $[\Delta;\lambda]$ and $[\Delta;\lambda]^*$ where

$$[\lambda]^* = \varepsilon \times [\lambda] \qquad \textit{and} \qquad [\Delta;\lambda]^* = \varepsilon \times [\Delta;\lambda] \tag{7.10}$$

Under $O(n) \downarrow SO(n)$ the distinction between an irreducible representation and its associate is lost. However, only those irreducible representations of $O(n)$ which are *NOT* self-associated remain irreducible under $O(n) \downarrow SO(n)$. Each self-associate irreducible representation of $O(2k)$ yields on restriction to $SO(2k)$ two inequivalent irreducible representations of the same dimension which we shall label as $[\lambda]_\pm$ and $[\Delta;\lambda]_\pm$ where in the former case λ is necessarily a partition into k non-zero parts.

Table 7.1 Standard labels for the irreducible representations of the classical groups of rank k.

Group G	Label $\lambda_{\rm G}$	Constraint
$U(n)$	$\{\lambda\}$	$\ell_\lambda \le n$
$SU(n)$	$\{\lambda\}$	$\ell_\lambda \le n-1$
$O(2k+1)$	$[\lambda], [\lambda]^*$	$\ell_\lambda \le k$
	$[\Delta;\lambda], [\Delta;\lambda]^*$	$\ell_\lambda \le k$
$SO(2k+1)$	$[\lambda]$	$\ell_\lambda \le k$
	$[\Delta;\lambda]$	$\ell_\lambda \le k$
$O(2k)$	$[\lambda], [\lambda]^*$	$\ell_\lambda < k$
	$[\lambda]$	$\ell_\lambda = k$
	$[\Delta;\lambda]$	$\ell_\lambda \le k$
$SO(2k)$	$[\lambda]$	$\ell_\lambda < k$
	$[\lambda]_+, [\lambda]_-$	$\ell_\lambda = k$
	$[\Delta;\lambda]_+, [\Delta;\lambda]_-$	$\ell_\lambda \le k$
$Sp(2k)$	$\langle\lambda\rangle$	$\ell_\lambda \le k$

■ 7.5 Modification rules

The labels given in Table 7.1 uniquely label the inequivalent irreducible representations of the classical groups. However in many practical applications non-standard labels may arise. In such cases the corresponding character may either vanish or be equal to the character of an irreducible representation specified by a G-standard label or be the negative of such a character.

All the classical groups modification rules can be associated with a common procedure. The key operation is the removal of a continuous boundary strip of boxes of length h from the Young diagram specified by the partition (λ), starting at the foot of the first column and ending in the $c-$th column, to yield symbolically $\lambda - h$. If the resaulting Young diagram is regular then $\lambda - h$ is simply the partition which serves to specify the diagram to which we associate a sign factor $(-1)^c$. If the resulting diagram is not regular then it is discarded since the character vanishes identically. The procedure is repeated until the diagram either corresponds to that of a standard label or vanishes.

Table 7.2. Modification rules for the classical groups. $(p = \ell_\lambda, q = \ell_\mu)$

$U(n), SU(n)$	$\{\bar{\mu};\lambda\} = (-1)^{c+d-1}\{\overline{\mu-h};\lambda-h\}$	$h = p+q-n-1 \geq 0$
$O(2k+1)$	$[\lambda] = (-1)^{c-1}[\lambda-h]^*$	$h = 2p-2k-1 > 0$
	$[\lambda]^* = (-1)^{c-1}[\lambda-h]$	$h = 2p-2k-1 > 0$
	$[\Delta;\lambda] = (-1)^{c}[\Delta;\lambda-h]^*$	$h = 2p-2k-2 \geq 0$
	$[\Delta;\lambda]^* = (-1)^{c}[\Delta;\lambda-h]$	$h = 2p-2k-2 \geq 0$
$SO(2k+1)$	$[\lambda] = (-1)^{c-1}[\lambda-h]$	$h = 2p-2k-1 > 0$
	$[\Delta;\lambda] = (-1)^{c}[\Delta;\lambda-h]$	$h = 2p-2k-2 \geq 0$
$O(2k)$	$[\lambda] = (-1)^{c-1}[\lambda-h]^*$	$h = 2p-2k > 0$
	$[\lambda]^* = (-1)^{c-1}[\lambda-h]$	$h = 2p-2k > 0$
	$[\Delta;\lambda] = (-1)^{c}[\Delta;\lambda-h]$	$h = 2p-2k-1 \geq 0$
$SO(2k)$	$[\lambda] = (-1)^{c-1}[\lambda-h]$	$h = 2p-2k > 0$
	$[\Delta;\lambda] = (-1)^{c}[\Delta;\lambda-h]$	$h = 2p-2k-1 \geq 0$
	$[\Box;\lambda] = (-1)^{c-1}[\Box;\lambda-h]$	$h = 2p-2k-2 \geq 0$
	$[\Delta;\lambda]_\pm = (-1)^{c}[\Delta;\lambda-h]_\mp$	$h = 2p-2k-1 \geq 0$
	$[\Box;\lambda]_\pm = (-1)^{c-1}[\Box;\lambda-h]_{-\mp}$	$h = 2p-2k-2 \geq 0$
$Sp(2k)$	$\langle\lambda\rangle = (-1)^{c}\langle\lambda-h\rangle$	$h = 2p-2k-2 \geq 0$

■ Exercises

7.1 Verify for $O(6)$ that

$$[3211] = [32]^*, \quad \textit{and} \quad [\Delta;3211] = -[\Delta;321]$$

7.2 Verify for $Sp(4)$ that

$$\langle 2^2 1^2\rangle = -\langle 2^2\rangle, \quad \textit{and} \quad \langle 1^6\rangle = -\langle 0\rangle$$

7.3 Verify for $SO(8)$ that

$$[32^2 1^2] = [32^2], \quad [32^2 1^3] = 0, \quad [32^2 1^4] = -[32]$$

■ 7.6 Note on Mixed tensor irreducible representations of $U(n)$

So far we have only discussed the covariant tensor irreducible representations of $U(n)$ which herein will be our principal concern. In addition to the covariant tensor irreducible representations $\{\lambda\}$ there are inequivalent irreducible representations associated with $m-$th rank contravariant tensors specified by $\{\bar{\mu}\}$ and more generally irreducible representations associated with mixed tensors specified by $\{\bar{\mu};\lambda\}$. The technical details are given in BKW.

Symmetric Functions and the Symmetric Group 8
B. G. Wybourne

If the reader thinks he is done, now, and that this book
has no moral to it, he is in error. The moral of it is
this: If you are of any account, stay at home and make your
way by faithful diligence; but if you are of "no account",
go away from home, and then you will *have* to work, whether
you want to or not. Thus you become a blessing to your friends by
ceasing to be a nuisance to them
Mark Twain *Roughing it* (1872).

■ 8.1 Some basic branching rules

The unitary group, $U(n)$, contains many subgroups of relevance to applications in physics. Time does not permit detailed derivations, these can be found in references given earlier. Three basic branching rules can be written symbolically in terms skews of the special infinite series $S-$functions B, D, M introduced earlier:-

$$U(n) \downarrow U(n-1) \qquad \{\lambda\} \downarrow \{\lambda/M\} \tag{8.1a}$$
$$U(n) \downarrow O(n) \qquad \{\lambda\} \downarrow [\lambda/D] \tag{8.1b}$$
$$U(2n) \downarrow Sp(2n) \qquad \{\lambda\} \downarrow \langle\lambda/B\rangle \tag{8.1c}$$

Recall

$$M = \{0\} + \{1\} + \{2\} + \{3\} + \{4\} + \{5\} + \{6\} + \{7\} + \{8\} + \dots \tag{8.2a}$$
$$D = \{0\} + \{2\} + \{2^2\} + \{2^3\} + \{2^4\} + \{4\} + \{42\} + \{42^2\} + \{4^2\} + \{6\} + \{62\} + \{8\} + \dots \tag{8.2b}$$
$$B = \{0\} + \{1^2\} + \{1^4\} + \{1^6\} + \{1^8\} + \{2^2\} + \{2^21^2\} + \{2^21^4\} + \{2^4\} + \{3^2\} + \{3^21^2\} + \{4^2\} + \dots \tag{8.2c}$$

Note that (8.1a-c) hold for all n the distinction comes only in the application of modification rules, if required. Thus from (8.1a) we obtain for $U(n) \downarrow U(n-1)$ the typical $n-$independent results

$$\{21\} \downarrow \{21\} + \{2\} + \{1^2\} + \{1\} \tag{8.3a}$$
$$\{31^2\} \downarrow \{31^2\} + \{31\} + \{21^2\} + \{21\} + \{1^3\} + \{1^2\} \tag{8.3b}$$
$$\{42\} \downarrow \{42\} + \{41\} + \{4\} + \{32\} + \{31\} + \{3\} + \{2^2\} + \{21\} + \{2\} \tag{8.3c}$$
$$\{32^2\} \downarrow \{32^2\} + \{321\} + \{32\} + \{2^3\} + \{2^21\} + \{2^2\} \tag{8.3d}$$

The above results hold without modification if $n \geq 4$. Actually, (8.3a) and (8.3c) hold for $n \geq 3$. For $n = 2$ we would need to discard the $S-$functions involving two parts appearing on the right-hand-side of (8.3a) and (8.3c) while (8.3b) and (8.3d) would be completely null.

The branching rule for $SU(n) \downarrow SU(n-1)$ is the same as in (8.1a) with the proviso that inequivalent irreducible representations of $SU(n)$ involve at most $n-1$ non-zero parts. Thus under $SU(3) \downarrow SU(2)$ we have

$$\{21\} \downarrow 2\{1\} + \{2\} + \{0\} \tag{8.4a}$$

which dimensionally corresponds to

$$\bar{8} \downarrow 2\ \underline{2} + \underline{3} + \underline{1} \tag{8.4b}$$

■ 8.2 Examples of $U(n) \downarrow O(n)$ branching rules

Use of (8.1b) readily leads to the typical $n-$independent results:-

$$\{21\} \downarrow [21] + [1] \tag{8.5a}$$
$$\{2^21^2\} \downarrow [2^21^2] + [21^2] + [1^2] \tag{8.5b}$$
$$\{32^21\} \downarrow [32^21] + [321] + [31] + [2^3] + [2^21^2] + [2^2] + [21^2] + [2] + [1^2] \tag{8.5c}$$

Recall that the tensor irreducible representations of $O(2k)$ and $O(2k+1)$ are labelled by partitions having at most k non-zero parts. Thus the above results would be valid without modification for $k \geq 4$. For smaller values of k the $O(n)$ modification rules must be applied. Thus for $U(6) \downarrow O(6)$ the above results would modify to

$$\{21\} \downarrow [21] + [1] \tag{8.6a}$$
$$\{2^2 1^2\} \downarrow [2^2]^* + [21^2] + [1^2] \tag{8.6b}$$
$$\{32^2 1\} \downarrow [321] + [31] + [2^3] + [2^2]^* + [2^2] + [21^2] + [2] + [1^2] \tag{8.6c}$$

since

$$[2211] \equiv [2^2]^*, \quad [32^2 1] \equiv 0, \tag{8.7}$$

The results for $SU(6) \downarrow SO(6)$ can be obtained from (8.6) by noting that under $O(2k) \downarrow SO(2k)$ we have

$$[\lambda]^\star \equiv [\lambda] \tag{8.8}$$

and if $[\lambda]$ has k non-zero parts then

$$[\lambda] \equiv [\lambda]_+ + [\lambda]_- \tag{8.9}$$

and hence for $SU(6) \downarrow SO(6)$ (8.6) becomes

$$\{21\} \downarrow [21] + [1] \tag{8.10a}$$
$$\{2^2 1^2\} \downarrow [2^2] + [21^2]_+ + [21^2]_- + [1^2] \tag{8.10b}$$
$$\{32^2 1\} \downarrow [321]_+ + [321]_- + [31] + [2^3]_+ + [2^3]_- + 2[2^2] + [21^2]_+ + [21^2]_- + [2] + [1^2] \tag{8.10c}$$

■ 8.3 Examples of $U(2n) \downarrow Sp(2n)$ branching rules

The $U(2n) \downarrow Sp(2n)$ branching rules follow from (8.1c) and use of the $B-$series leads to the typical results:-

$$\{21\} \downarrow \langle 21\rangle + \langle 1\rangle \tag{8.11a}$$
$$\{2^2 1^2\} \downarrow \langle 2^2 1^2\rangle + \langle 2^2\rangle + \langle 21^2\rangle + \langle 1^4\rangle + 2\langle 1^2\rangle + \langle 0\rangle \tag{8.11b}$$
$$\{32^2 1\} \downarrow \langle 32^2 1\rangle + \langle 321\rangle + \langle 31^3\rangle + \langle 31\rangle + \langle 2^3\rangle + \langle 2^2 1^2\rangle + \langle 2^2\rangle + 2\langle 21^2\rangle + \langle 2\rangle + \langle 1^2\rangle \tag{8.11c}$$

For $n \geq 4$ the above results are $n-$independent. For smaller values of n the modification rules must be applied to give for $SU(6) \downarrow Sp(6)$ the modified results:-

$$\{21\} \downarrow \langle 21\rangle + \langle 1\rangle \tag{8.12a}$$
$$\{2^2 1^2\} \downarrow \langle 2^2\rangle + \langle 21^2\rangle + 2\langle 1^2\rangle + \langle 0\rangle \tag{8.12b}$$
$$\{32^2 1\} \downarrow \langle 321\rangle + \langle 31\rangle + \langle 2^3\rangle + \langle 2^2\rangle + 2\langle 21^2\rangle + \langle 2\rangle + \langle 1^2\rangle \tag{8.12c}$$

In the above cases the modification rules make the terms that have four non-zero parts null. This is generally not the case.

■ 8.4 A branching rule theorem

Suppose that G ⊃ H and that the vector irreducible representation 1_{G} branches as

$$1_{\mathrm{G}} \downarrow \sum_\nu \nu_{\mathrm{H}} \tag{8.13}$$

with ν_{H} not necessarily irreducible, then

$$\lambda_{\mathrm{G}} \downarrow (\sum_\nu \nu_{\mathrm{H}}) \otimes \lambda_{\mathrm{G}} \tag{8.14}$$

(I gave this result long ago B G Wybourne *Symmetry Principles in Atomic Spectroscopy* New York - Wiley-Interscience (1970).) This result is very useful in obtaining new branching rules as we now show.

■ 8.5 The $U(m+n) \downarrow U(m) \times U(n)$ branching rule

In this case the vector irreducible representation $\{1\}$ of $U(m+n)$ branches as

$$\{1\} \downarrow \{1\}_m + \{1\}_n \tag{8.15}$$

and from (8.14)

$$\{\lambda\}_{m+n} \downarrow (\{1\}_m + \{1\}_n) \otimes \{\lambda\} \quad (8.16a)$$
$$= \sum_\zeta (\{1\}_m \otimes \{\lambda/\zeta\}) \times (\{1\}_n \otimes \{\zeta\}) \quad (8.16b)$$
$$= \sum_\zeta \{\lambda/\zeta\}_m \times \{\zeta\}_n \quad (8.16c)$$

where in (8.16b) we have used (6.12d) and in (8.16c) used the fact that

$$\{1\} \otimes \{\mu\} \equiv \{\mu\} \quad (8.17)$$

■ 8.6 Example

Consider the irreducible representation {21} of some group-subgroup $U(m+n) \downarrow U(m) \times U(n)$. From (8.16c) we have

$$\{21\} \downarrow \sum_\zeta \{21/\zeta\}_m \times \{\zeta\}_n \quad (8.18)$$

The sum is over all $S-$functions $\{\zeta\}$ that skew {21} i.e. the set

$$\{\zeta\} = \{21\} + \{2\} + \{1^2\} + \{1\} + \{0\} \quad (8.19)$$

using those in (8.18) leads to the branching rule

$$\{21\} \downarrow \{21\} \times \{0\} + \{2\} \times \{1\} + \{1^2\} \times \{1\} + \{1\} \times \{2\} + \{1\} \times \{1^2\} + \{0\} \times \{21\} \quad (8.20)$$

The above results also hold for $SU(m+n) \downarrow SU(m) \times SU(n)$ with the usual partition provisos. Thus for $SU(7) \downarrow SU(4) \times SU(3)$ we can check that (8.20) is dimensionally correct by noting that

$$\underline{112} = \underline{20} \times \underline{1} + \underline{10} \times \underline{3} + \underline{6} \times \underline{3} + \underline{4} \times \underline{6} + \underline{4} \times \underline{3} + \underline{1} \times \underline{8} \quad (8.21)$$

■ 8.7 The $U(mn) \downarrow U(m) \times U(n)$ branching rule

In this case the vector irreducible representation branches as

$$\{1\}_{mn} \downarrow \{1\}_m \times \{1\}_n \quad (8.22)$$

and noting (6.12f) and (8.17) we obtain

$$\{\lambda\}_{mn} \downarrow (\{1\}_m \times \{1\}_n) \otimes \{\lambda\} \quad (8.23a)$$
$$= \sum_\rho (\{1\}_m \otimes \{\rho\}) \times (\{1\}_n \otimes \{\lambda \circ \rho\}) \quad (8.23b)$$
$$= \sum_\rho \{\rho\}_m \times \{\lambda \circ \rho\}_n \quad (8.23c)$$

Note the appearance of an inner $S-$function product $\{\lambda \circ \rho\}_n$. If $\{\lambda\} = \{1^k\}$ then recall that

$$\{1^k \circ \rho\} = \begin{cases} \{\rho\}' & \text{if } w_\rho = k \\ 0 & \text{if } w_\rho \neq k \end{cases} \quad (8.24)$$

where the partition $(\rho)'$ is the conjugate of the partition (ρ) while if $\{\lambda\} = \{k\}$ then

$$\{k \circ \rho\} = \begin{cases} \{\rho\} & \text{if } w_\rho = k \\ 0 & \text{if } w_\rho \neq k \end{cases} \quad (8.25)$$

Thus for the special case where $\{\lambda\} = \{k\}$ *or* $\{1^k\}$ we have the simplifications of (8.23), namely,

$$\{k\} \downarrow \sum_\rho \{\rho\}_m \times \{\rho\}_n \quad (8.26a)$$
$$\{1^k\} \downarrow \sum_\rho \{\rho\}_m \times \{\rho\}_n{}' \quad (8.26b)$$

where in both cases the summation over ρ is restricted to partitions (ρ) of weight k. Equation (8.26a) is appropriate to k identical bosons and (8.26b) to k identical fermions.

■ 8.8 Classification of the states of the d^N electron configurations

A $d-$orbital has spin $S = \frac{1}{2}$ and orbital angular momentum $L = 2$. Thus there are 10 spin-orbital states that can be regarded as forming a basis for the vector irreducible representation $\{1\}$ of the special unitary group $SU(10)$. the spin and orbital parts of the wave function will span the direct product subgroup $SU(2) \times SU(5)$ subgroup of $SU(10)$. If we have N electrons in equivalent $d-$orbitals (same principal quantum number n) to satisfy the Pauli exclusion principle their wavefunction must be totally antisymmetric with respect to the spin-orbital quantum numbers. This will be the case if they span the irreducible representation $\{1^N\}$ of $SU(10)$. Thus to determine the total spin S and orbital L quantum numbers we need to first use (8.26b) to determine the

$$SU(10) \downarrow SU(2)^S \times SU(5)^L \tag{8.27}$$

branching rules for the $\{1^N\}$ irreducible representation of $SU(10)$. Inequivalent irreducible representations of $SU(2)$ are labelled by single non-negative integers while irreducible representations of the type $\{p, q\}$ are associated with the equivalence

$$\{p, q\} \equiv \{p - q\} \qquad p \geq q \geq 0 \tag{8.28}$$

An irreducible representation $\{p\}$ of $SU(2)$ will be of dimension

$$dimension(\{p\}) = p + 1 \tag{8.29}$$

and spin

$$spin(\{p\}) = S = \frac{p}{2} \tag{8.30}$$

Thus from (8.26b) we have under $SU(10) \downarrow SU(2) \times SU(5)$

$$\{1^N\} \downarrow \sum_{\substack{\sigma_1 \geq \sigma_2 \geq 0 \\ \sigma_1 + \sigma_2 = N}} \{\sigma_1, \sigma_2\} \times \{\sigma_1, \sigma_2\}' \tag{8.31a}$$

$$= \sum_{\substack{\sigma_1 \geq \sigma_2 \geq 0 \\ \sigma_1 + \sigma_2 = N}} \{\sigma_1 - \sigma_2\} \times \{2^{\sigma_2} 1^{\sigma_1 - \sigma_2}\} \tag{8.31b}$$

where we have made use of the fact that $SU(2)$ limits (σ) in (8.26b) to two row partitions and the conjugate partition must then be a two column partition. Below we give a table of the branching rules for $N = 0, 1, \ldots, 10$.

Table 8.1 Branching rules for the $\{1^N\}$ irreducible representations under $SU(10) \downarrow SU(2) \times SU(5)$

Dim	$SU(10) \downarrow$	$SU(2) \times SU(5)$
1	$\{0\}$	$\{0\} \times \{0\}$
10	$\{1\}$	$\{1\} \times \{1\}$
45	$\{1^2\}$	$\{2\} \times \{1^2\} + \{0\} \times \{2\}$
120	$\{1^3\}$	$\{3\} \times \{1^3\} + \{1\} \times \{21\}$
210	$\{1^4\}$	$\{4\} \times \{1^4\} + \{2\} \times \{21^2\} + \{0\} \times \{2^2\}$
252	$\{1^5\}$	$\{5\} \times \{0\} + \{3\} \times \{21^3\} + \{1\} \times \{2^21\}$
210	$\{1^6\}$	$\{4\} \times \{1\} + \{2\} \times \{2^21^2\} + \{0\} \times \{2^3\}$
120	$\{1^7\}$	$\{3\} \times \{1^2\} + \{1\} \times \{2^31\}$
45	$\{1^8\}$	$\{2\} \times \{1^3\} + \{0\} \times \{2^4\}$
10	$\{1^9\}$	$\{1\} \times \{1^4\}$
1	$\{1^{10}\}$	$\{0\} \times \{0\}$

(8.32)

Notice that the partitions labelling the irreducible representations of $SU(5)$ involve at most 4 non-zero integers. The single integers labelling the irreducible representations of $SU(2)$ are twice the value of the spin S and all the states belonging to the associated $SU(5)$ irreducible representation have that spin quantum number. Further, notice that the dimensions of the $SU(10)$ irreducible representations $\{1^N\}$ and $\{1^{10-N}\}$ are equal and that the $SU(2)$ spins are the same with each $SU(5)$ being the contragredient partner. This is the familiar particle-hole symmetry manifesting itself.

To proceed further we need to branch the $SU(5)$ irreducible representations into those of its subgroup $SO(5)$ using (8.1b). We note that the branchings for contragredient partners are identical so that it suffices to consider just those irreducible representations of $SU(5)$ that occur in Table 8.1 for $N \leq 5$.

Table 8.2 Branching rules for $SU(5) \downarrow SO(5)$.

Dim	$SU(5)$	$\downarrow SO(5)$
1	$\{0\}$	$[0]$
5	$\{1\}$	$[1]$
10	$\{1^2\}$	$[1^2]$
15	$\{2\}$	$[2]+[0]$
10	$\{1^3\}$	$[1^2]$
40	$\{21\}$	$[21]+[1]$
5	$\{1^4\}$	$[1]$
45	$\{21^2]$	$[21]+[1^2]$
50	$\{2^2\}$	$[2^2]+[2]+[0]$
1	$\{1^5\}$	$[0]$
24	$\{21^3\}$	$[2]+[1^2]$
75	$\{2^21\}$	$[2^2]+[21]+[1]$

(8.33)

Finally, to complete the classification of the states of d^N we need the branching rules for $SO(5) \downarrow SO(3)$. These can be found by use of the branching rule theorem of (8.14). Since under $SO(5) \downarrow SO(3)$

$$[1] \downarrow [2] \tag{8.34}$$

we have from (8.14) that

$$[\lambda] \downarrow [2] \otimes [\lambda] \tag{8.35a}$$

$$= [((\{2\} - \{0\}) \otimes \{\lambda/C\})/D] \tag{8.35b}$$

In going from (8.35a) to (8.35b) we have made use of the possibility of inverting the $SU(5) \downarrow SO(5)$ branching rule by use of the $C-$series of $S-$functions that is the inverse of the $D-$series. Thus in going from (8.35a) to (8.35b) we have replaced [2] by $\{2/C\} = \{2\} - \{0\}$ and $[\lambda]$ by $\{\lambda/C\}$ and then evaluated the plethysms as for $S-$functions and then skew the resultant list of $S-$functions with the $D-$series and applied the $SO(3)$ modification rules to reduce everything to a list of single part $SO(3)$ labels and given them their standard spectroscopic angular momentum labels L where the correspondence is

$$[0]S, [1]P, [2]D, [3]F, [4]G, [5]H, [6]I, [7]K, [8]L, [9]M, [10]N, [11]O, [12]Q, \ldots \tag{8.36}$$

Table 8.3 Some $SO(5) \downarrow SO(3)$ branching rules.

Dim	$SO(5)$	$\downarrow SO(3)$
1	$[0]$	S
5	$[1]$	D
10	$[1^2]$	$P+F$
14	$[2]$	$D+G$
35	$[21]$	$P+D+F+G+H$
35	$[2^2]$	$S+D+F+G+I$

(8.37)

Thus we have all the components to classify the states of the d^N electron configurations using the group-subgroup scheme

$$SU(10) \supset SU(2)^S \times (SU(5) \supset SO(5) \supset SO(3)^L) \tag{8.38}$$

Table 8.4 Group classification of the states of the d^N electron configurations.

d^N	$SU(2)\times SU(5)$	$SO(5)$	$SO(3)$	^{2S+1}L
d^0	$\{0\}\times\{0\}$	$[0]$	$[0]$	^1S
d^1	$\{1\}\times\{1\}$	$[1]$	$[1]$	^2D
d^2	$\{2\}\times\{1^2\}$	$[1^2]$	$[1]+[3]$	^3PF
	$\{1^2\}\times\{2\}$	$[2]$	$[2]+[4]$	^1DG
		$[0]$	$[0]$	^1S
d^3	$\{3\}\times\{1^3\}$	$[1^2]$	$[1]+[3]$	^4PF
	$\{1\}\times\{21\}$	$[21]$	$[1]+[2]+[3]+[4]+[5]$	^2PDFGH
		$[1]$	$[2]$	^2D
d^4	$\{4\}\times\{1^4\}$	$[1]$	$[2]$	^5D
	$\{2\}\times\{21^2\}$	$[21]$	$[1]+[2]+[3]+[4]+[5]$	^3PDFGH
		$[1]$	$[2]$	^3D
	$\{0\}\times\{2^2\}$	$[2^2]$	$[0]+[2]+[3]+[4]+[6]$	^1SDFGI
		$[2]$	$[2]+[4]$	^1DG
		$[0]$	$[0]$	^1S
d^5	$\{5\}\times\{0\}$	$[0]$	$[0]$	^6S
	$\{3\}\times\{21^3\}$	$[2]$	$[0]+[2]$	^4SD
		$[1^2$	$[1]+[3]$	^4PF
	$\{1\}\times\{2^21\}$	$[2^2]$	$[0]+[2]+[3]+[4]+[6]$	^2SDFGI
		$[21]$	$[1]+[2]+[3]+[4]+[5]$	^2PDFGH
		$[1]$	$[2]$	^2D

(8.39)

Note that every state has a distinct set of labels. This would not be the case if we had simply enumerated the ^{2S+1}L states.

■ 8.9 Seniority classification of the states of d^N

We could have used several other possible subgroup structures embedded in $SU(10)$ to classify the states of d^N. Let us consider the group-subgroup structure

$$SU(10)\downarrow Sp(10)\downarrow SU(2)\times(SO(5)\downarrow SO(3)) \tag{8.40}$$

The $SU(10)\downarrow Sp(10)$ branching rules follow from (8.1c) to give the results in Table 8.5.

Table 8.5 Some $SU(10)\downarrow Sp(10)$ branching rules.

$SU(10)\downarrow$	$Sp(10)$
$\{0\}$	$\langle 0\rangle$
$\{1\}$	$\langle 1\rangle$
$\{1^2\}$	$\langle 1^2\rangle+\langle 0\rangle$
$\{1^3\}$	$\langle 1^3\rangle+\langle 1\rangle$
$\{1^4\}$	$\langle 1^4\rangle+\langle 1^2\rangle+\langle 0\rangle$
$\{1^5\}$	$\langle 1^5\rangle+\langle 1^3\rangle+\langle 1\rangle$

(8.41)

The branching rules for $Sp(2k)\downarrow SU(2)\times SO(2k+1)$ follow from the branching rule theorem to give generally

$$\langle\lambda\rangle\downarrow\sum_{\sigma}\{\lambda/A\circ\sigma\}\times[\sigma/D] \tag{8.42}$$

The evaluation may be readily done in SCHUR either using the branching rule 11 or writing the simple function

```
gr sp10
enter rv1
dim[rv1]
gr2su2so5
rule[rv1*0]sk1with a
rule last sum i1eq2
rule last sk2with d
supout false
std last
dim last
stop
```

Running the function gives for example

```
DP>
fn1
Group is Sp(10)
enter rv1
1111
Dimension = 165
Groups are   SU(2) * SO(5)
      {4}[1] + {2}[21] + {0}[2^2 ]
Dimension = 165
DP>
```

Thus we obtain the $Sp(10) \downarrow SU(2) \times SO(5)$ branching rules given in Table 8.6.

Table 8.6 Some $Sp(10) \downarrow SU(2) \times SO(5)$ branching rules.

Dim	$Sp(10) \downarrow$	$SU(2) \times SO(5)$
1	$\langle 0 \rangle$	$\{0\} \times [0]$
10	$\langle 1 \rangle$	$\{1\} \times [1]$
44	$\langle 1^2 \rangle$	$\{2\} \times [1^2] + \{0\} \times [2]$
110	$\langle 1^3 \rangle$	$\{3\} \times [1^2] + \{1\} \times [21]$
165	$\langle 1^4 \rangle$	$\{4\} \times [1] + \{2\} \times [21] + \{0\} \times [2^2]$
132	$\langle 1^5 \rangle$	$\{5\} \times [0] + \{3\} \times [2] + \{1\} \times [2^2]$

(8.43)

Combining the results of Tables 8.5 and 8.6 with that of 8.3 gives the classification of the d^N states shown in Table 8.7.

Table 8.7 The symplectic classification of the states of the d^N electron configurations.

d^N	$SU(10)\downarrow$	$Sp(10)\downarrow$	$SU(2)\times SO(5)\downarrow$	^{2S+1}L
d^0	$\{0\}$	$\langle 0\rangle$	$\{0\}\times[0]$	^1S
d^1	$\{1\}$	$\langle 1\rangle$	$\{1\}\times[1]$	^2D
d^2	$\{1^2\}$	$\langle 1^2\rangle$	$\{2\}\times[1^2]$	^3PF
			$\{0\}\times[2]$	^1DG
		$\langle 0\rangle$	$\{0\}\times[0]$	^1S
d^3	$\{1^3\}$	$\langle 1^3\rangle$	$\{3\}\times[1^2]$	^4PF
			$\{1\}\times[21]$	^2PDFGH
		$\langle 1\rangle$	$\{1\}\times[1]$	^2D
d^4	$\{1^4\}$	$\langle 1^4\rangle$	$\{4\}\times[1]$	^5D
			$\{2\}\times[21]$	^3PDFGH
			$\{0\}\times[2^2]$	^1SDFGI
		$\langle 1^2\rangle$	$\{2\}\times[1^2]$	^3PF
			$\{0\}\times[2]$	^1DG
		$\langle 0\rangle$	$\{0\}\times[0]$	^1S
d^5	$\{1^5\}$	$\langle 1^5\rangle$	$\{5\}\times[0]$	^6D
			$\{3\}\times[2]$	^4DG
			$\{1\}\times[2^2]$	^2SDFGI
		$\langle 1^3\rangle$	$\{3\}\times[1^2]$	^4PF
			$\{1\}\times[21]$	^2PDFGH
		$\langle 1\rangle$	$\{1\}\times[1]$	^2D (8.44)

A careful inspection of the above table reveals a striking property, known as *seniority*.In d^2 we note that the ^1S state has the same $Sp(10)$ label as does the corresponding state for d^0. It is as if in going from d^0 to d^2 two $d-$electrons have paired to produce an angular momentum state $L=0$. Let the integer v be the value of N for which the $Sp(10)$ irreducible representation $\langle 1^N\rangle$ first occurs then $\frac{N-v}{2}$ is the number of pairs of $d-$electrons coupled to zero angular momentum in forming the $N-$particle state. For example $\langle 1^2\rangle$ occurs in d^2, d^4 and those states are assigned seniority $v=2$. Seniority is a useful concept in calculating matrix elements in atomic physics and of much greater usefulness in nuclear shell calculations where strong pairing interactions occur. In nuclei states of lowest energy have lowest seniority whereas in atomic shells one has the opposite situation.

Symmetric Functions and the Symmetric Group 9
B. G. Wybourne

Mathematics requires a small dose, not of genius, but of imaginative freedom which, in a larger dose, would be insanity. And if mathematicians tend to burn out early in their careers, it is probably because life has forced them to acquire too much common sense, thereby rendering them too sane to work. But by then they are sane enough to teach, so a use can still be found for them.
— Angus K Rodgers

■ 9.1 Quantum Dots and Symmetry Physics

The subject of quantum dots involves the confinement of N electrons in two or three dimensions, commonly by electrostatic fields, over a nano-metre scale. The confining potential is, to a good approximation parabolic. The quantum dot behaves as an $N-$electron atom without a nuclear core. One may add or subtract a single electron from a quantum dot giving rise to the possibility of nano-metre scale devices such as transitors etc.

In an atom the kinetic energy tends to dominate over the potential energy (the confinement length is small) whereas in a quantum dot the two contributions are roughly of the same order making normal perturbative methods difficult. A closely analogous problem is that of nucleons confined in a harmonic oscillator potential with quantised motion occuring about the centre of mass of the $N-$nucleon system. We shall first review some of the properties of the isotropic harmonic oscillator, the unitary group $U(3)$ and the special unitary group $SU(3)$.

■ 9.2 The Isotropic harmonic oscillator

The Hamiltonian H of a normalised isotropic harmonic oscillator (i.e. with $m = \hbar = \omega = 1$) in three-dimensions may be written as

$$H = \frac{1}{2}(\mathbf{p}^2 + \mathbf{r}^2) \tag{9.1}$$

From Heisenberg's quantisation postulate the coordinates q_i and momenta p_i satisfy the commutation relations

$$[q_i, q_j] = [p_i, p_j] = 0, \qquad [q_i, p_j] = i\delta_{ij} \tag{9.2}$$

Now introduce boson annihilation and creation operators (a and $\mathrm{a}^\dagger$ respectively)

$$\mathbf{a} = \frac{1}{\sqrt{2}}(\mathbf{r} + i\mathbf{p}), \qquad \mathbf{a}^\dagger = \frac{1}{\sqrt{2}}(\mathbf{r} - i\mathbf{p}) \tag{9.3}$$

which satisfy the bosonic commutation relation

$$[a_i, a_j^\dagger] = \delta_{ij} \tag{9.4}$$

The Hamiltonian can now be written as

$$H = \mathbf{a}^\dagger \cdot \mathbf{a} + \frac{3}{2} \tag{9.5}$$

Use of Eqn. (9.4) then leads to

$$[H, a_j^\dagger] = a_j^\dagger, \qquad [H, a_j] = -a_j \tag{9.6}$$

Thus we deduce that $a_j^\dagger$ creates and a_j annihilates a quantum in the j direction. We recognise $\mathbf{a}^\dagger \cdot \mathbf{a}$ as being the *number operator* with eigenvalues of

$$n = n_1 + n_2 + n_3 \tag{9.7}$$

and hence the energy eigenvalues of H are

$$E_n = n + \frac{3}{2} \quad (n = 0, 1, 2, \ldots) \tag{9.8}$$

with normalised state vectors

$$|n_1 n_2 n_3\rangle = \prod_{i=1}^{3} \frac{a_i^{\dagger n_i}}{\sqrt{n_i!}} |000\rangle \tag{9.9}$$

with $|000\rangle$ being the vacuum state with

$$a_j|000\rangle = 0 \tag{9.10}$$

Noting that $\mathrm{a}^\dagger = \mathrm{a}^*$ we have

$$\langle n_1 n_2 n_3| = \langle 000| \prod_{i=1}^{3} \frac{a_i^{n_i}}{\sqrt{n_i!}} \tag{9.11}$$

with

$$\langle 000|a_j^\dagger = 0 \tag{9.12}$$

■ 9.3 Degeneracy Group of the Isotropic Harmonic Oscillator

Let us introduce nine operators

$$T_{ij} = \frac{1}{2}\{a_i^\dagger, a_j\} \qquad (i, j = 1, 2, 3) \tag{9.13}$$

where $\{a, b\} \equiv ab + ba$. Using the basic boson commutation relations of Eqn. (9.4) we find

$$[T_{ij}, T_{rs}] = \delta_{jr} T_{is} - \delta_{is} T_{rj} \tag{9.14}$$

Thus the nine operators T_{ij} close under commutation and generate a Lie algebra. Putting $H_i \equiv T_{ii}$ (do not confuse this with the Hamiltonian) we find the three H_i form a self-commuting set and

$$[H_i, T_{jr}] = (\delta_{ij} - \delta_{ir}) T_{jr} \tag{9.15}$$

all the roots are of the form $e_i - e_j$ where the e are mutually orthogonal unit vectors.

The set of nine operators T_{ij} may be identified as the generators of the unitary group in three dimensions, $U(3)$. The Hamiltonian H is related to the H_i of Eqn. (9.15) via

$$H = H_1 + H_2 + H_3 \tag{9.16}$$

commutes with all T_{ij}. The three operators

$$H' = H_i - \frac{H}{3} \tag{9.17}$$

taken with the T_{ij} $(i \neq j)$ can be taken as the generators of the special unitary group $SU(3)$ if we remember that since $\sum_i H_i' = 0$ the H_i' are not linearly independent. For reasons that will become apparent shortly we refer to $U(3)$ as the degeneracy group of the isotropic harmonic oscillator.

■ 9.4 Labelling Representations and Weights

In the case of the angular momentum group $SO(3)$ we label the angular momentum states as $|JM\rangle$ where M is the eigenvalue of J_z with J being the *highest weight* of M. This idea carries over to Lie groups in general. We recall that in the case of $SO(3)$ we can write the defining commutation relations as

$$[J_z, J_\pm] = \pm L_\pm \qquad [J_+, J_-] = J_z \tag{9.17}$$

with

$$J_\pm = \frac{1}{\sqrt(2)}(J_x \pm iJ_y) \tag{9.18}$$

For a general semisimple Lie algebra of rank ℓ we have ℓ operators ,H_i $(i = 1, \ldots, \ell)$, that commute among themselves. The Lie algebra can be cast into the standard Cartan-Weyl form as

$$\begin{aligned}
[H_i, H_j] &= 0 \qquad (i, j = 1, \ldots \ell) \\
[H_i, E_\alpha] &= \alpha_i E_\alpha \\
[E_\alpha, E_\beta] &= N_{\alpha\beta} E_{\alpha+\beta} \\
[E_\alpha, E_{-\alpha}] &= \alpha^i H_i
\end{aligned} \tag{9.19}$$

where the E_α are the analogues of the ladder operators $J_\pm$ of SO_3.

Just as in SO_3 we distinguish the components of a representation by the eigenvalues of J_z for a Lie group we may label the components of a representation by the eigenvalues of the ℓ self-commuting operators H_i. For any compact Lie algebra the *highest* weight vector is unique and hence can be used to specify the representation. Consider for example, the group $U(3)$ which has three self-commuting operators H_i. Suppose we wish to determine the representation of $U(3)$ whose components are the annihilation a and creation operators $a^\dagger$, we have

$$[H_i, a_j^\dagger] = \delta_{ij} a_j^\dagger \quad \text{and} \quad [H_i, a_j] = -\delta_{ij} a_j \tag{9.20}$$

Thus the components of $\mathbf{a}^\dagger$ give rise to the set of weight vectors $(100), (010), (001)$. The highest weight vector is (100) and hence we can label the representation as $\{100\}$ of $U(3)$. Likewise, the components of a give rise to the weight vectors $(-100), (0-10), (00-1)$. We say that a weight vector w is higher than a weight vector w' if the first component of their difference $w - w'$ is *positive*. Thus the highest weight for a is $(00-1)$ and the representation of $U(3)$ spanned by the components of a may be labelled as $\{00-1\}$ which is *contragredient* to $\{100\}$.

■ Exercises

9.1 Noting Eqn(9.14) show that the nine operators T_{ij} are associated with the nine weight vectors (000), (000), (000), (1-10), (10-1), (01-1), (-110), (-101), (0-11).

9.2 Determine the highest weight vector in the above set of weight vectors.

9.3 Repeat the above analysis for a two-dimensional isotropic harmonic oscillator and show that the relevant symmetry group is $U(2)$.

■ 9.5 Rotational Symmetry and the Isotropic Harmonic Oscillator

The harmonic oscillator Hamiltonian, Eqn. (9.1), commutes with all the components of the angular momentum operator

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = i\mathbf{a} \times \mathbf{a}^\dagger \tag{9.21}$$

and hence H is rotationally invariant. The components of $\mathbf{L}$ form under commutation the Lie algebra associated with the group $SO(3)$. Noting the definition of the operators T_{ij}, Eqn.(9.13), and Eqn. (9.21) we have

$$L_1 = -i(T_{23} - T_{32}), \quad L_2 = -i(T_{31} - T_{13}), \quad L_3 = -i(T_{12} - T_{21}) \tag{9.22}$$

We may choose L_3 as the generator of the group $SO(2)$ and hence for the three-dimensional isotropic harmonic oscillator we have the group structure

$$U(3) \supset SU(3) \supset SO(3) \supset SO(2) \tag{9.23}$$

It is convenient to label the oscillator states in a basis $|n\ell m\rangle$ where $n = 0, 1, 2, \ldots$. We have

$$n = 2x + \ell \qquad \text{with} \quad x = 0, 1, 2, \ldots \tag{9.23}$$

and hence the values of ℓ associated with a given value of n are

$$\begin{aligned} \ell &= 1, 3, 5, \ldots, n \qquad n \quad odd \\ &= 0, 2, 4, \ldots, n \qquad n \quad even \end{aligned} \tag{9.24}$$

and thus for a given n there is a set of $\frac{(n+1)(n+2)}{2}$-fold degenerate states $|n\ell m\rangle$. This is precisely the dimension of the symmetric representation of $U(3)$ designated by the partition $\{n, 0, 0\}$ and hence the statement that the group $U(3)$ is the *degeneracy group* of the three-dimensional isotropic harmonic oscillator.

$n = 5$	————————	p, f, h
$n = 4$	————————	s, d, g
$n = 3$	————————	p, f
$n = 2$	————————	s, d
$n = 1$	————————	p
$n = 0$	————————	s

The first six levels of the isotropic harmonic oscillator

In the preceding we have developed the theory for a *single* particle in a harmonic oscillator potential. This particle could equally well be a nucleon as in nuclear physics or an electron in a quantum dot. The degeneracies are exactly the same as is the form of the energy spectrum. To proceed further requires we develop a many-particle model for particles interacting in a harmonic oscillator potential. To that end we may seek to develop a *dynamical group*.

Two combinatorial observations

These notes are supplementary to Symmetric Functions 9 and concern

1. Some additional remarks on boson-fermion symmetry.
2. An observation relating to the Littlewood-Richardson Rule.

1. Boson-Fermion symmetry for a one-dimensional harmonic oscillator

The $n-$ dimensional isotropic harmonic oscillator has the metaplectic group $Mp(2n)$ as its dynamical group with $U(n)$ as the degeneracy group. The complete set of states span the infinite dimensional unitary irreducible representation $\tilde{\Delta}$ of $Mp(2n)$. Under $Mp(2n) \to U(n)$ one has the branching rule

$$\tilde{\Delta} \to M \tag{1}$$

where

$$M = \sum_{m=0}^{\infty} \{m\} \tag{2}$$

Consider N non-interacting particles in an $n-$dimensional harmonic oscillator potential. In general these particles will form states belonging to symmetrised powers (or plethysms) according to the various partitions of the integer N i.e. with respect to the group $U(n)$ terms coming from the plethysm

$$M \otimes \{\lambda\} \tag{3}$$

Now consider the special case of $n = 1$ with either N bosons or fermions. The degeneracy group is now just $U(1)$ and we can readily evaluate Eq.(3) for the totally symmetric and totally antisymmetric cases as plethysms. At the $U(1)$ level we have for the $M-$series

$$M \otimes \{N\} = \sum_k g_N^k \{k\} \tag{4}$$

where g_N^k is the number of partitions of k into at most N parts with repetitions and null parts allowed and

$$M \otimes \{1^N\} = \sum_\ell c_N^\ell \{\ell\} \tag{5}$$

where c_N^ℓ is the number of partitions of ℓ into N distinct parts, including the null part.

If $\ell = k + (N^2 - N)/2$ then we have the identity

$$c_N^\ell = g_N^k \tag{6}$$

For example,

$$M \otimes \{4\} \supset \{0\} + \{1\} + 2\{2\} + 3\{3\} + 5\{4\} + 6\{5\} + 9\{6\} + 11\{7\} + 15\{8\} + 18\{9\} \tag{7}$$

$$M \otimes \{1^4\} \supset \{6\} + \{7\} + 2\{8\} + 3\{9\} + 5\{10\} + 6\{11\} + 9\{12\} + 11\{13\} + 15\{14\} + 18\{15\} \tag{8}$$

$$c_N^{13} = g_N^7$$

For g_4^7 and c_4^{13} we have the respective sets of 11 partitions

$$g_4^7\{2^31\} + \{321^2\} + \{32^2\} + \{3^21\} + \{41^3\} + \{421\} + \{43\} + \{51^2\} + \{52\} + \{61\} + \{7\}$$

$$c_4^{13}\{5431\} + \{6421\} + \{643\} + \{652\} + \{7321\} + \{742\} + \{751\} + \{832\} + \{841\} + \{931\} + \{1021\}$$

The identity, Eq.(6), comes about by realising that one can map from one of the sets of partitions to the other by adding or subtracting $\rho_N = (N-1, \ldots, 2, 1, 0)$. Adding ρ to the partitions of k into at most N parts, converts them into partitions, all of whose parts are distinct. Hence $c^\ell = g^k if \ell = k + \frac{1}{2}N(N-1)$. Thus in the example above add $(3, 2, 1, 0)$ to the g_4^7 list gives that of c_4^{13}.

The consequence of the boson-fermion equivalence is that the thermodynamic properties of $N-$non-interacting bosons or fermions are essentially equivalent apart from a shift in the groundstate.

2. Littlewood-Richardson coefficients

Kirillov has noted that if $c^{\lambda}_{\mu\nu} = 1$ then

$$c^{N\lambda}_{N\mu,N\nu} = 1 \tag{1}$$

His observation can be conjectured to generalise to

$$c^{N\lambda}_{N\mu,N\nu} = \binom{N+k-1}{k-1} \quad if \quad c^{\lambda}_{\mu\nu} = k \tag{2}$$

where in both cases N multiplies all the parts of the attached partition. e.g.

$$\{321\} \cdot \{431\} \supset$$
$$4\{24\ 20\ 84\} + 4\{24\ 16\ 12\ 4\} + 4\{24\ 16\ 84^2\} + 4\{20\ 16\ 12\ 8\} + 4\{20\ 16\ 12\ 4^2\} + 4\{20\ 16\ 8^24\} \tag{3}$$

and

$$\{12\ 84\} \cdot \{16\ 12\ 4\} \supset$$
$$35\{24\ 20\ 84\} + 35\{24\ 16\ 12\ 4\} + 35\{24\ 16\ 84^2\} + 35\{20\ 16\ 12\ 8\} + 35\{20\ 16\ 12\ 4^2\} + 35\{20\ 16\ 8^24\} \tag{4}$$

This looks encouraging BUT there exist counterexamples! One counter example is worth billions of examples!

Symmetric Functions and the Symmetric Group 10 B. G. Wybourne

Oh, he seems like an okay person, except for being a little strange in some ways. All day he sits at his desk and scribbles, scribbles. scribbles. Then at the end of the day, he takes the sheets of paper he's scribbled on, scrunges them all up, and throws them in the trash can
— J von Neumann's housekeeper

10.1 A Hamiltonian for Quantum Dots

Experimentally the electrons of a quantum dot are contained in a parabolic potential and hence we expect a close relationship with a many-electron system subject to a harmonic oscillator potential. The interaction potential $V(r_i, r_j)$ between particles i and j moving in a two-dimensional confining potential in the $x-y$ plane is taken to saturate at small particle separations and to decrease quadratically with increasing separation. In free space we would expect the interaction between two electrons to vary as $|r_i - r_j|^{-1}$. In a quantum dot the form of $V(r_i, r_j)$ is modified by the presence of image charges. The wavefunctions of the electrons confined in the quantum dots have a small but finite extent in the $z-$direction perpendicular to the $x-y$ plane. This results in a smearing of the electron charges along the $z-$direction. As a result the interparticle repulsion tends to saturate at small distances. This suggests choosing the interaction as

$$V(r_i, r_j) = 2V_0 - \frac{1}{2}m^*\Omega^2|r_i - r_j|^2 \tag{10.1}$$

where m^* is the electron effective mass and V_0 and Ω are positive parameters.

Consider an $N-$electron quantum dot each with a charge $-e$, a $g-$factor g^*, spatial coordinates r_i and spin components $s_{z,i}$ along the $z-$axis. Suppose there is a magnetic field B along the $z-$axis. The spatial part of the Hamiltonian can be written as

$$H_{space} = \frac{1}{2m^*}\sum_i \left[p_i + \frac{eA_i}{c}\right]^2 + \frac{1}{2}m^*\omega_0^2\sum_i |r_i|^2 + \sum_{i<j} V(r_i, r_j) \tag{10.2}$$

and the spin part as

$$H_{spin} = -g^*\mu_B B\sum_i s_{z,i} \tag{10.3}$$

where the momentum and vector potential associated with the $i-th$ electron are given by

$$p_i = (p_{x,i}, p_{y,i}) \qquad A_i = (A_{x,i}, A_{y,i}) \tag{10.4}$$

and μ_B is the Bohr magneton.

The eigenstates of H will involve the product of the spatial and spin eigenstates obtained from $H_{spatial}$ and H_{spin}. The total spin projection $S_Z = \sum_i s_{z,i}$ will be a good quantum number. Choosing a circular gauge $A_i = B(-y_i/2, x_i/2, 0)$ Eqn. (10.2) becomes

$$H_{space} = \frac{1}{2m^*}\sum_i p_i^2 + \frac{1}{2}m^*\omega_0^2(B)\sum_i |r_i|^2 + \sum_{i<j}\left[2V_0 - \frac{1}{2}m^*\Omega^2|r_i, r_j|^2\right] + \frac{\omega_c}{2}\sum_i L_{z,i} \tag{10.5}$$

where $\omega_0^2(B) = \omega_0^2 + \omega_c^2/4$ and $\omega_c = eB/m^*c$.

10.2 Note on Commutators and Second-quantisation

In much that follows we will need to be able to manipulate bosonic annihilation (a_i) and creation operators $(a_i^\dagger)$. The basic bosonic commutation relations are

$$[a_i, a_j] = 0, \quad [a_i^\dagger, a_j^\dagger] = 0, \quad [a_i, a_j^\dagger] = \delta_{i,j} \tag{10.6}$$

These can be used to simplify expressions. As an example, consider the anticommutator $\{a_i^\dagger, a_j\} = a_i^\dagger a_j + a_j a_i^\dagger$ and let us evaluate the commutator $[\{a_i^\dagger, a_j\}, a_k]$. Expanding out we have

$$[a_i^\dagger a_j + a_j a_i^\dagger, a_k] = [a_i^\dagger a_j, a_k] + [a_j a_i^\dagger, a_k] \tag{10.7}$$

Expanding out the first commutator we have

$$[a_i^\dagger a_j, a_k] = a_i^\dagger a_j a_k - a_k a_i^\dagger a_j \tag{10.8}$$

To simplify this commutator we want to try to rearrange the first term on the right-hand-side to cancel the second term. Using the first commutator in Eqno. (10.6) we can rearrange the first term as

$$a_i^\dagger a_j a_k \to a_i^\dagger a_k a_j \tag{10.9}$$

and hence the right-hand-side of Eqn. (10.9) becomes

$$\begin{aligned} a_i^\dagger a_j a_k - a_k a_i^\dagger a_j &\to a_i^\dagger a_k a_j - a_k a_i^\dagger a_j \\ &= [a_i^\dagger, a_k] a_j \\ &= -[a_k, a_i^\dagger] a_j \\ &= -\delta_{i,k} a_j \end{aligned}$$

■ **Exercise**

Show that if

$$T_{ij} = \frac{1}{2}\{a_i^\dagger, a_j\}$$

then

$$[T_{ij}, T_{rs}] = \delta_{j,r} T_{is} - \delta_{i,s} T_{rj}$$

■ 10.3 The Degeneracy Group for Mesoscopic Systems

In this lecture we enlarge the concept of a *degeneracy* group to a *dynamical* group. The degeneracy group for the isotropic harmonic oscillator was found to be $SU(3)$. Each irreducible representation $\{n00\}$ is spanned by a set of $\frac{(n+1)(n+2)}{2}$ eigenstates of the Hamiltonian and associated with the *same* energy eigenvalue E_n of the harmonic oscillator. There is one weight vector for every eigenstate. The algebra of the degeneracy group contains a set of operators that allow us to start from any eigenstate and ladder through the entire set of degenerate eigenstates associated with a given degenerate eigenvalue. Thus the angular momentum ladder operators $L_\pm$ take us from one $|\alpha LM\rangle$ eigenstate to another $|\alpha LM \pm 1\rangle$ but leaving L fixed. The operators $L_z, L_\pm$ that generate the angular momentum group SO_3 but cannot take us from states belonging to one irreducible representation of SO_3 to another. To do that we must use the operators contained in the degeneracy algebra that lie outside of those of the angular momentum algebra. In addition the algebra of the degeneracy group contains operators that allow us to ladder between states of a given $SU(3)$ multiplet changing *both* L and M quantum numbers but *not* n. These additional operators reflect the fact that the isotropic harmonic oscillator has, like the $H-$atom, symmetry higher that just rotational symmetry.

■ 10.4 A Dynamical Group

We seek a *dynamical* group that contains the degeneracy group as a subgroup and has the energy eigenstates belonging to a single irreducible representation. Such a group contains among its generators operators that allow one to ladder between different irreducible representations of the degeneracy group. The degeneracy group contains an infinite set of finite dimensional unitary irreducible representations and hence the dynamical group must necessarily be a non-compact group with infinite dimensional unitary irreducible representations . We now construct the dynamical group for mesoscopic quantum systems.

■ 10.5 The Dynamical Group for Mesoscopic Quantum Systems

1. Assume the Hamiltonian of the $N-$particle system is a function of coordinate and momentum operators of the individual particles.

2. Designate the coordinates of the $r-$th particle by x_{ri} with $r = 1, \ldots, N$ and the momentum by p_{ri} with $i = 1, \ldots, d$.
3. The associated operators X_{ri} and P_{ri} obey the usual Heisenberg commutation relations (We choose units such that $\hbar = 1$)

$$[X_{ri}, X_{sj}] = 0, [X_{ri}, P_{sj}] = i\delta_{rs}\delta_{ij}, [P_{ri}, P_{sj}] = 0 \tag{10.10}$$

4. The $(2Nd)^2$ bilinear operators

$$\{X_{ri}X_{sj}, X_{ri}P_{sj}, P_{ri}X_{sj}, P_{ri}P_{sj}\} \tag{10.11}$$

close under commutation. However, only
$(2Nd+1)Nd$ of these operators are independent since

$$P_{ri}X_{sj} = X_{sj}P_{ri} - i\delta_{rs}\delta_{ij} \tag{10.12}$$

5. Consider the $(2Nd+1)Nd$ independent operators

$$Q_{risj} = \frac{1}{2}\{X_{ri}, X_{sj}\}, \quad V_{risj} = \frac{1}{2}\{X_{ri}, P_{sj}\},$$
$$K_{risj} = \frac{1}{2}\{P_{ri}, P_{sj}\} \tag{10.13}$$

They close under commutation on the non-
compact Lie algebra $Sp(2Nd, R)$ which we can take as the dynamical algebra of our mesoscopic $N-$ electron system.

■ 10.6 Subalgebras of the Dynamical Algebra

1. We can construct subalgebras of $Sp(2Nd, R$ by forming subsets of the defining generators that close under commutation. Thus, for example, the V's close under commutation forming the elements of the $GL(Nd, R)$ algebra.
2. Contracting on particle or spatial indices can yield further Lie subalgebras. Thus the two sets of operators (summing on repeated indices)

$$Q_{ij} = X_{ri}X_{rj}, \quad L_{ij} = X_{ri}P_{rj} - X_{rj}P_{ri},$$
$$K_{ij} = P_{ri}P_{rj}$$
$$T_{ij} = \frac{1}{2}(X_{ri}P_{rj} + X_{rj}P_{ri} + P_{ri}X_{rj} + P_{rj}X_{ri}) \tag{10.14}$$

and

$$Q_{rs} = X_{ri}X_{si}, \quad L_{rs} = X_{ri}P_{si} - X_{si}P_{ri},$$
$$K_{rs} = P_{ri}P_{si}$$
$$T_{rs} = \frac{1}{2}(X_{ri}P_{si} + X_{si}P_{ri} + P_{ri}X_{si} + P_{si}X_{ri}) \tag{10.15}$$

close under commutation and separately generate the Lie algebras $Sp(2d, R)$ and $Sp(2N, R)$.
3. The above two algebras do not commute but the subsets $\{L_{ij}\}$ and $\{L_{rs}\}$ do separately close under commutation with

$$[L_{ij}, L_{kl}] = i(L_{ik}\delta_{jl} - L_{il}\delta_{jk} + L_{jk}\delta_{il} + L_{jk}\delta_{il} - L_{jl}\delta_{ik})$$
$$[L_{rs}, L_{tu}] = i(L_{rt}\delta_{su} - L_{ru}\delta_{st} + L_{st}\delta_{ru} - L_{su}\delta_{rt}) \tag{10.16}$$

and form the generators of the subalgebras $O(d)$ and $O(N)$.
4. Continuing we are led to the following possible Lie subalgebras of $Sp(2Nd, R)$:-

$$Sp(2, R) \times O(Nd) \supset Sp(2, R) \times O(N) \times O(d)$$
$$\supset U(1) \times O(N) \times O(d) \tag{10.17}$$

$$Sp(2N, R) \times O(d) \supset U(N) \times O(d) \supset U(1) \times O(N) \times O(d) \tag{10.18}$$

$$Sp(2d) \times O(N) \supset U(d) \times O(N) \supset U(1) \times O(d) \times O(N) \tag{10.19}$$

$$U(Nd) \supset U(N) \times U(d) \supset U(1) \times O(N) \times O(d) \tag{10.20}$$

Note the separation of the spatial and particle dependencies.

■ 10.7 Identification of the $Sp(2,R)$ Subgroup

Let us introduce three operators defined by

$$Q = X_{ri}X_{ri}, \quad T = X_{ri}P_{ri} + P_{ri}X_{ri}, \quad K = P_{ri}P_{ri} \tag{10.21}$$

and having the non-zero commutation relations

$$[Q,K] = 2iT, \quad [Q,T] = 4iQ, \quad [K,T] = -4iK \tag{10.21}$$

These commutation relations are those of a three element Lie algebra. Let us first decide if the algebra is compact or non-compact. This we may do by calculating the metric tensor

$$g_{ij} = c_{ik}^{t} c_{jt}^{k} \tag{10.22}$$

where the c_{ik}^{t} are the structure constants of the Lie algebra. Noting Eqn. (10.21) we have

$$c_{QK}^{T} = 2i, \quad c_{QT}^{Q} = 4i, \quad c_{KT}^{K} = -4i \tag{10.23}$$

Recall that the structure constants are antisymmetric. We now find for the diagonal elements of the metric tensor

$$\begin{aligned} g_{QQ} &= g_{KK} = 0 \\ g_{TT} &= c_{TQ}^{Q} c_{TQ}^{Q} + c_{TK}^{K} c_{TK}^{K} = -4i \times -4i + 4i \times 4i = -32 \end{aligned} \tag{10.24}$$

In addition we have the off-diagonal elements

$$g_{QK} = g_{KQ} = c_{QT}^{Q} c_{KQ}^{T} + c_{QK}^{T} c_{KT}^{K} = 4i \times -2i + 2i \times -4i = 16 \tag{10.25}$$

and thus the complete metric tensor is represented by the matrix

$$[g_{ij}] = \begin{array}{c} \\ Q \\ K \\ T \end{array} \begin{array}{c} \begin{array}{ccc} Q & K & T \end{array} \\ \begin{pmatrix} 0 & 16 & 0 \\ 16 & 0 & 0 \\ 0 & 0 & -32 \end{pmatrix} \end{array} \tag{10.26}$$

We can produce a diagonal metric tensor by putting

$$A_{\pm} = \frac{1}{\sqrt{2}}(Q \pm K) \tag{10.27}$$

to give the Lie algebra as

$$[A_{\pm}, T] = 4iA_{\mp}, \qquad [A_{+}, A_{-}] = 2iT \tag{10.28}$$

and the metric tensor as

$$[g_{ij}] = \begin{array}{c} \\ A_{+} \\ A_{-} \\ T \end{array} \begin{array}{c} \begin{array}{ccc} A_{+} & A_{-} & T \end{array} \\ \begin{pmatrix} -16 & 0 & 0 \\ 0 & +16 & 0 \\ 0 & 0 & -32 \end{pmatrix} \end{array} \tag{10.29}$$

We first note that the metric tensor has $\det|g_{ij}| \not= 0$ and hence we can conclude that the Lie algebra is semisimple. Furthermore the metric tensor is indefinite as required for the algebra to correspond to be non-compact. and hence our Lie algebra is necessarily

$$SO(2,1) \sim Sp(2,R) \tag{10.30}$$

■ 10.8 Back to the Quantum Dot Hamiltonian

We can express terms in the Hamiltonian of an isotropic harmonic oscillator

$$H_o = \frac{1}{2m} P_{ri}P_{ri} + \frac{m\omega^2}{2} X_{ri}X_{ri} \tag{10.31}$$

in terms of the group generators of $Sp(2, R)$ by noting that

$$\frac{1}{2m} P_{ri} P_{ri} = \frac{1}{2m} K \tag{10.31}$$

and

$$\frac{m\omega^2}{2} X_{ri} X_{ri} = \frac{m\omega^2}{2} Q \tag{10.32}$$

to give

$$H_o = \frac{1}{2m} K + \frac{m\omega^2}{2} Q \tag{10.33}$$

Now consider our earlier Hamiltonian

$$H_{space} = \frac{1}{2m^*} \sum_i p_i^2 + \frac{1}{2} m^* \omega_0^2(B) \sum_i |r_i|^2 + \sum_{i<j} \left[2V_0 - \frac{1}{2} m^* \Omega^2 |r_i, r_j|^2 \right] + \frac{\omega_c}{2} \sum_i L_{z,i} \tag{10.5}$$

We can write the electron-electron interaction term for an $N-$electron quantum dot as

$$N(N-1)V_0 - \frac{m\Omega^2}{4} \sum_{rsi} (X_{ri} - X_{si})(X_{ri} - X_{si})$$

leading to

$$H_{space} = \frac{1}{2m} K + \frac{m\Omega_0^2}{2} Q - \frac{eB}{4mc} L_{12} + N(N-1)V_0 + \frac{m\Omega^2}{2} \sum_{rs} Q_{rs} \tag{10.34}$$

with

$$\Omega_0^2 = \omega^2 + (\frac{eB}{2mc})^2 - N\Omega^2 \tag{10.35}$$

The significance of these results is that the first three terms in Eqno. (10.34) have been expressed in terms of the generators of $Sp(2, R)$ (K, Q) and $O(d)$ (L_{12}) and the last term in terms of generators of the group $Sp(2N, R)$. A practical calculation then involves the evaluation of matrix elements of the group generators in a harmonic oscillator basis.

Symmetric Functions and the Symmetric Group 11
B. G. Wybourne

It was a dark and stormy night when R. H. Bing volunteered to drive some stranded mathematicians from the fogged-in Madison airport to Chicago. Freezing rain pelted the windscreen and iced the roadway as Bing drove on - concentrating deeply on the mathematical theorem he was explaining. Soon the windshield was fogged from the energetic explanation. The passengers too had beaded brows, but their sweat rose from fear. As the mathematical description got brighter, the visibility got dimmer. Finally, the conferees felt a trace of hope for their survival when Bing reached forward - apparently to wipe off the moisture from the windshield. Their hope turned to horror when, instead, Bing drew a figure with his finger on the foggy pane and continued his proof - embellishing the illustration with arrows and helpful labels as needed for the demonstration.
— Prof. R H Bing, famous US topologist

■ 10.1 Introduction

Today I want to remark on properties of bosons and fermions. Recall that bosons are objects with integer spin and that the wavefunction for $N-$identical bosons is totally symmetric while fermions are objects with half-integer spin and that the wavefunction for $N-$identical fermions is totally antisymmetric. Note the use of the word *totally.* Here we will be considering identical fermions, or bosons, in an isotropic harmonic oscillator. The $N-$particle wavefunction is the product of a spin part with a spatial part and symmetrization involves both parts. Thus a $N-$particle wavefunction maybe totally antisymmetric (*fermions*) or totally symmetric (*bosons*) yet the spin or spatial parts need not be, their product *must* be. I will start by going through the analysis of the isotropic harmonic oscillator for a single particle and then sketch the problem of enumerating states for the case of $N-$noninteracting particles - bosons or fermions. Much of what we have to say in this lecture is applicable to nuclei, quantum dots and statistical physics.

■ 10.2 The isotropic three-dimensional harmonic oscillator

For a one-dimensional harmonic oscillator one has the well-known energy spectrum (putting $\hbar = 1$)

$$E_n = (n + \tfrac{1}{2})\nu \qquad n = 0, 1, \ldots \tag{1}$$

where ν is the usual frequency. We can regard an isotropic three-dimensional harmonic oscillator as three one-dimensional harmonic oscillator each of frequency ν and energy eigenvalues

$$E_n = (n + \tfrac{3}{2})\nu \qquad n = 0, 1, \ldots \tag{2}$$

with

$$n = n_x + n_y + n_z \qquad n_x, n_y, n_z = 0, 1, \ldots \tag{3}$$

Each state may be labelled by the triplet of numbers (n_x, n_y, n_z) and for a given value of n we will obtain $(n+1)(n+2)/2$ distinct triplets (n_x, n_y, n_z). For example, the first four levels are associated with the states given below and corresponding to degeneracies, 1, 3, 6, 10. The triplets of quantum numbers can be regarded as the *weights* of the irreducible representations $\{n\}$ of the degeneracy group $U(3)$. As the isotropic three-dimensional harmonic oscillator is clearly rotationally invariant we could extend our description of our states by using the group chain

$$U(3) \supset SO(3) \supset SO(2) \tag{4}$$

and uniquely label the states by the set of quantum numbers $|n\ell m_\ell\rangle$, noting that for n odd and even we have the $U(3) \to SO(3)$ decompositions

$$n \to [n] + [n-2] + \ldots + \begin{cases} [0] & \text{if } n \text{ is even} \\ [1] & \text{if } n \text{ is odd} \end{cases} \tag{5}$$

Table 1. The values of the triplets (n_x, n_y, n_z) for the first four levels of an isotropic three-dimensional harmonic oscillator.

n	(n_x, n_y, n_z)
0	$(0,0,0)$
1	$(1,0,0)$
	$(0,1,0)$
	$(0,0,1)$
2	$(2,0,0)$
	$(0,2,0)$
	$(0,0,2)$
	$(1,1,0)$
	$(1,0,1)$
	$(0,1,1)$
3	$(3,0,0)$
	$(0,3,0)$
	$(0,0,3)$
	$(2,1,0)$
	$(2,0,1)$
	$(0,2,1)$
	$(0,1,2)$
	$(1,2,0)$
	$(1,0,2)$
	$(1,1,1)$

So far we have ignored the spin of our single particle. It is the spin of our particle that determines whether we are considering fermions or bosons. A complete description of the one-particle states requires the complete set of quantum numbers $|sm_s n\ell m_\ell\rangle$. This increases the degeneracies by a factor of $(2s+1)$ and extends the degeneracy group to $SU(2) \times U(3)$. Let us now look carefully at the case of two identical noninteracting fermions or bosons in a one-dimensional harmonic oscillator.

Two-particles in a one-dimensional harmonic oscillator

Let us suppose we have a boson with spin $s = 0$ and a fermion with spin $s = \frac{1}{2}$. Placed in a one-dimensional harmonic oscillator we have an infinite set of equi-spaced energy levels that may be indexed by integers $m = 0, 1, \ldots$. Each level has a spatial degeneracy of 1 and a spin degeneracy of $2s+1$. The degeneracy group is $SU(2) \times U(1)$ with each level labelled as ${}^{2s+1}\{m\}$ with m labelling the one-dimensional irreducible representations of $U(1)$. The complete set of one-particle states span the infinite set of $SU(2) \times U(1)$ irreducible representations that may be succinctly written as

$$\{2s\} \times M, \tag{6}$$

where

$$M = \sum_{m=0}^{\infty} \{m\}. \tag{7}$$

Now consider we place two noninteracting bosons (or fermions) in a one-dimensional harmonic oscillator. The bosonic two-particle states must be totally symmetric and the fermionic states totally antisymmetric with respect to the spin and spatial variables.

For the bosons the states all have $S = 0$ and the spatial symmetries come from extracting the symmetric part of $M \times M$ under $U(1)$ to give

$$\{M \times M\}_{sym} = \sum_{n=0}^{\infty} g^n \{n\}, \tag{8}$$

where

$$g^n = \left[\frac{n}{2}\right] + 1, \tag{9}$$

with $\left[\frac{n}{2}\right]$ is the integer part of $n/2$. Thus the degeneracy of the two-particle boson state labelled as n will be g^n.

Now consider the two-fermion states. These will have spin $S = 0$ (spin singlets) or $S = 1$ (spin triplets). At the $U(1)$ level the spatial part for the spin singlets will involve the symmetric part of $M \times M$ and hence give rise to the same states as in Eq.(8) while for the spin triplets the spatial part will involve the antisymmetric part of $M \times M$ with

$$\{M \times M\}_{anti} = \sum_{p=1}^{\infty} c^p\{p\}, \tag{10}$$

where

$$c^p = \left[\frac{p-1}{2}\right] + 1. \tag{11}$$

Clearly,

$$c^p = g^{p-1}. \tag{12}$$

Thus there is a one-to-one correspondence (relative to their respective ground states) between multiplicities of the $S = 0$ two-particle states of a pair of identical bosons or fermions. Likewise there is a one-to-one correspondence between the multiplicities of the $S = 0$ states of a pair of identical bosons and those of the $S = 1$ states of a pair of identical fermions shifted according to Eq.(12).

Results of (8) and (10) follow directly from

K Grudzinski and B G Wybourne, J. Phys. A:Math.Gen.**29**,6631 (1996).

N noninteracting particles in a one-dimensional harmonic oscillator

For $N-$noninteracting bosons in a one-dimensional harmonic oscillator one simply enumerates the $U(1)$ content of the symmetric part of the $N-$fold product $M^{\otimes\{N\}}$ to find that

$$M^{\otimes\{N\}} = \sum_{k=0}^{\infty} g_N^k\{k\} \tag{13}$$

where g_N^k is the number of partitions of k into at most N parts allowing repetitions and null parts. For example,

$$\begin{aligned} M^{\otimes\{4\}} \supset & \{0\} + \{1\} + 2\{2\} + 3\{3\} + 5\{4\} + 6\{5\} + 9\{6\} + 11\{7\} \\ & + 15\{8\} + 18\{9\} + 23\{10\} + \dots \end{aligned} \tag{14}$$

Now consider $N-$noninteracting fermions. To describe the multiplicities of the levels we need the antisymmetric $SU(2) \times U(1)$ content of

$$(\{1\} \times M)^{\otimes\{1^N\}} = \sum_{\sigma \vdash N} \{\sigma\} \times M^{\otimes\{\sigma'\}} \tag{15}$$

where the sum is over all partitions $(\sigma) = (\sigma_1, \sigma_2)$ of N into at most two parts with (σ') being the partition conjugate to (σ), involving partitions whose Young frame involves at most two columns. Thus for $N = 4$ we would have

$$(\{1\} \times M)^{\otimes\{1^4\}} = \{4\} \times M^{\otimes\{1^4\}} + \{31\} \times M^{\otimes\{21^2\}} + \{2^2\} \times M^{\otimes\{2^2\}} \tag{16}$$

The spin S_σ to be associated with a given partition (σ_1, σ_2) is

$$S_\sigma = \frac{\sigma_1 - \sigma_2}{2} \tag{17}$$

Under $U(1)$ the tensor products, say $\{p\} \times \{q\}$, have the simple form

$$\{p\} \times \{q\} = \{p + q\} \tag{18}$$

Using this fact one can show that under $U(1)$

$$M^{\otimes\{1^N\}} = \sum_{\ell=\frac{N(N-1)}{2}}^{\infty} c_N^\ell \{\ell\} \tag{19}$$

where c_N^ℓ is the number of partitions of the integer ℓ into N distinct parts including the null part. In fact

$$c_N^\ell = g_N^k \qquad \text{if} \quad \ell = k + \tfrac{N(N-1)}{2} \tag{20}$$

This can be seen by noting that we can map the sets of partitions into the other by adding, or subtracting, $\rho_N = (N-1, \ldots, 2, 1, 0)$. Adding ρ_N to the partitions of k into at most N parts converts them into partitions, all of whose parts are distinct.

Thus in a one-dimensional harmonic oscillator there is a one-to-one correspondence between the multiplicities of the states of N identical bosons and those of the maximal spin states of N identical fermions. For four identical fermions we obtain the $U(1)$ content for the states with $S = 2$

$$\begin{aligned} M^{\otimes\{1^4\}} \supset & \{6\} + \{7\} + 2\{8\} + 3\{9\} + 5\{10\} + 6\{11\} + 9\{12\} \\ & + 11\{13\} + 15\{14\} + 18\{15\} + 23\{16\} + \ldots \end{aligned} \tag{21}$$

which may be compared with Eq.(14).

However, the spatially antisymmetric states are not the complete set of states for N noninteracting fermions, one also has the mixed symmetry states associated with the other spin states. For example, for $N = 4$ we also have the $U(1)$ states with spin $S = 1$ coming from

$$\begin{aligned} M^{\otimes\{21^2\}} \supset & \{3\} + 2\{4\} + 4\{5\} + 6\{6\} + 10\{7\} + 14\{8\} + 20\{9\} \\ & + 26\{10\} + 35\{11\} + 44\{12\} + 56\{13\} + 68\{14\} \\ & + 84\{15\} + 100\{16\} + \ldots \end{aligned} \tag{22}$$

and for the spin $S = 0$ states

$$\begin{aligned} M^{\otimes\{2^2\}} \supset & \{2\} + \{3\} + 3\{4\} + 4\{5\} + 7\{6\} + 9\{7\} + 14\{8\} + 17\{9\} \\ & + 24\{10\} + 29\{11\} + 38\{12\} + 45\{13\} + 57\{14\} \\ & + 66\{15\} + 81\{16\} + \ldots \end{aligned} \tag{23}$$

N noninteracting particles in an isotropic three-dimensional harmonic oscillator

In this case the degeneracy group is $SU(2) \times U(3)$. Let us consider a single fermion of spin $s = \frac{1}{2}$. The spin spans the $\{1\}$ irreducible representation of $SU(2)$ while the orbital states span the irreducible representations $\{m\}$ of $U(3)$ with $m = 0, 1, \ldots, \infty$. Thus for a single fermion the complete set of states span the reducible representation $\{1\} \times M$ of $SU(2) \times U(3)$ with

$$M = \sum_{m=0}^{\infty} \{m\} \tag{24}$$

Let us set the groundstate energy to zero and assume the levels are equi-spaced by an energy ΔE. For N noninteracting particles in an isotropic three-dimensional harmonic oscillator we obtain the totally antisymmetric states, under $SU(2) \times U(3)$,

$$(\{1\} \times M) \otimes \{1^N\} = \sum_{m \geq n \geq 0, m+n=N}^{N} \{m-n\} \times (M \otimes \{2^n 1^{m-n}\}) \tag{25}$$

$$= \sum_{m \geq n \geq 0, m+n=N}^{N} {}^{(m-n+1)}(M \otimes \{2^n 1^{m-n}\}) \tag{26}$$

$$= \sum_{S_{min}}^{\frac{N}{2}} {}^{2S+1}(M \otimes \{2^{\frac{N}{2}+S} 1^{2S}\}) \tag{27}$$

where

$$S_{min} = \begin{cases} \frac{1}{2} & \text{if } N \text{ is odd} \\ 0 & \text{if } N \text{ is even} \end{cases} \tag{28}$$

The spin multiplicity, $(2S+1)$, is given as a left superscript in (26) and (27).

For four identical, noninteracting, fermions of spin $\frac{1}{2}$ we obtain, to weight 10 in $U(3)$ the states:-

$$\begin{array}{lllll}
{}^1\{2\} & +\,2\,{}^1\{21\} & +\,{}^1\{21^2\} & +\,3\,{}^1\{2^2\} & +\,2\,{}^1\{2^21\} \\
+\,{}^1\{2^3\} & +\,{}^1\{3\} & +\,3\,{}^1\{31\} & +\,3\,{}^1\{31^2\} & +\,5\,{}^1\{32\} \\
+\,6\,{}^1\{321\} & +\,3\,{}^1\{32^2\} & +\,2\,{}^1\{3^2\} & +\,5\,{}^1\{3^21\} & +\,3\,{}^1\{3^22\} \\
+\,{}^1\{3^3\} & +\,3\,{}^1\{4\} & +\,7\,{}^1\{41\} & +\,5\,{}^1\{41^2\} & +\,12\,{}^1\{42\} \\
+\,12\,{}^1\{421\} & +\,8\,{}^1\{42^2\} & +\,11\,{}^1\{43\} & +\,14\,{}^1\{431\} & +\,12\,{}^1\{432\} \\
+\,5\,{}^1\{43^2\} & +\,11\,{}^1\{4^2\} & +\,11\,{}^1\{4^21\} & +\,12\,{}^1\{4^22\} & +\,4\,{}^1\{5\} \\
+\,11\,{}^1\{51\} & +\,10\,{}^1\{51^2\} & +\,19\,{}^1\{52\} & +\,22\,{}^1\{521\} & +\,13\,{}^1\{52^2\} \\
+\,21\,{}^1\{53\} & +\,30\,{}^1\{531\} & +\,24\,{}^1\{532\} & +\,24\,{}^1\{54\} & +\,32\,{}^1\{541\} \\
+\,14\,{}^1\{5^2\} & +\,7\,{}^1\{6\} & +\,18\,{}^1\{61\} & +\,15\,{}^1\{61^2\} & +\,32\,{}^1\{62\} \\
+\,36\,{}^1\{621\} & +\,24\,{}^1\{62^2\} & +\,39\,{}^1\{63\} & +\,52\,{}^1\{631\} & +\,48\,{}^1\{64\} \\
+\,9\,{}^1\{7\} & +\,25\,{}^1\{71\} & +\,23\,{}^1\{71^2\} & +\,45\,{}^1\{72\} & +\,54\,{}^1\{721\} \\
+\,59\,{}^1\{73\} & +\,14\,{}^1\{8\} & +\,37\,{}^1\{81\} & +\,32\,{}^1\{81^2\} & +\,67\,{}^1\{82\} \\
+\,17\,{}^1\{9\} & +\,48\,{}^1\{91\} & +\,24\,{}^1\{10\ \} & +\,{}^3\{1^2\} & +\,{}^3\{1^3\} \\
+\,2\,{}^3\{21\} & +\,3\,{}^3\{21^2\} & +\,{}^3\{2^2\} & +\,2\,{}^3\{2^21\} & +\,{}^3\{3\} \\
+\,5\,{}^3\{31\} & +\,6\,{}^3\{31^2\} & +\,6\,{}^3\{32\} & +\,8\,{}^3\{321\} & +\,3\,{}^3\{32^2\} \\
+\,7\,{}^3\{3^2\} & +\,9\,{}^3\{3^21\} & +\,7\,{}^3\{3^22\} & +\,4\,{}^3\{3^3\} & +\,2\,{}^3\{4\} \\
+\,9\,{}^3\{41\} & +\,11\,{}^3\{41^2\} & +\,12\,{}^3\{42\} & +\,17\,{}^3\{421\} & +\,7\,{}^3\{42^2\} \\
+\,16\,{}^3\{43\} & +\,23\,{}^3\{431\} & +\,17\,{}^3\{432\} & +\,11\,{}^3\{43^2\} & +\,10\,{}^3\{4^2\} \\
+\,16\,{}^3\{4^21\} & +\,12\,{}^3\{4^22\} & +\,4\,{}^3\{5\} & +\,16\,{}^3\{51\} & +\,18\,{}^3\{51^2\} \\
+\,24\,{}^3\{52\} & +\,32\,{}^3\{521\} & +\,16\,{}^3\{52^2\} & +\,34\,{}^3\{53\} & +\,48\,{}^3\{531\} \\
+\,38\,{}^3\{532\} & +\,32\,{}^3\{54\} & +\,48\,{}^3\{541\} & +\,26\,{}^3\{5^2\} & +\,6\,{}^3\{6\} \\
+\,24\,{}^3\{61\} & +\,28\,{}^3\{61^2\} & +\,38\,{}^3\{62\} & +\,52\,{}^3\{621\} & +\,27\,{}^3\{62^2\} \\
+\,56\,{}^3\{63\} & +\,82\,{}^3\{631\} & +\,60\,{}^3\{64\} & +\,10\,{}^3\{7\} & +\,36\,{}^3\{71\} \\
+\,40\,{}^3\{71^2\} & +\,60\,{}^3\{72\} & +\,80\,{}^3\{721\} & +\,90\,{}^3\{73\} & +\,14\,{}^3\{8\} \\
+\,50\,{}^3\{81\} & +\,56\,{}^3\{81^2\} & +\,85\,{}^3\{82\} & +\,20\,{}^3\{9\} & +\,69\,{}^3\{91\} \\
+\,26\,{}^3\{10\ \} & +\,{}^5\{1^3\} & +\,{}^5\{21^2\} & +\,{}^5\{31\} & +\,3\,{}^5\{31^2\} \\
+\,{}^5\{32\} & +\,2\,{}^5\{321\} & +\,2\,{}^5\{3^2\} & +\,4\,{}^5\{3^21\} & +\,2\,{}^5\{3^22\} \\
+\,3\,{}^5\{3^3\} & +\,2\,{}^5\{41\} & +\,5\,{}^5\{41^2\} & +\,3\,{}^5\{42\} & +\,5\,{}^5\{421\} \\
+\,{}^5\{42^2\} & +\,5\,{}^5\{43\} & +\,9\,{}^5\{431\} & +\,5\,{}^5\{432\} & +\,5\,{}^5\{43^2\} \\
+\,3\,{}^5\{4^2\} & +\,5\,{}^5\{4^21\} & +\,3\,{}^5\{4^22\} & +\,3\,{}^5\{51\} & +\,8\,{}^5\{51^2\} \\
+\,5\,{}^5\{52\} & +\,10\,{}^5\{521\} & +\,3\,{}^5\{52^2\} & +\,9\,{}^5\{53\} & +\,18\,{}^5\{531\} \\
+\,12\,{}^5\{532\} & +\,8\,{}^5\{54\} & +\,16\,{}^5\{541\} & +\,6\,{}^5\{5^2\} & +\,{}^5\{6\} \\
+\,6\,{}^5\{61\} & +\,11\,{}^5\{61^2\} & +\,10\,{}^5\{62\} & +\,16\,{}^5\{621\} & +\,6\,{}^5\{62^2\} \\
+\,17\,{}^5\{63\} & +\,29\,{}^5\{631\} & +\,18\,{}^5\{64\} & +\,{}^5\{7\} & +\,9\,{}^5\{71\} \\
+\,17\,{}^5\{71^2\} & +\,15\,{}^5\{72\} & +\,26\,{}^5\{721\} & +\,26\,{}^5\{73\} & +\,2\,{}^5\{8\} \\
+\,13\,{}^5\{81\} & +\,22\,{}^5\{81^2\} & +\,23\,{}^5\{82\} & +\,3\,{}^5\{9\} & +\,18\,{}^5\{91\} \\
+\,5\,{}^5\{10\ \} & +\ldots & & &
\end{array} \tag{29}$$

Terms having $U(3)$ irreducible representations, $\{\lambda\}$, of the same weight, $|\lambda|$, will have the same energy, E_λ, relative to the groundstate and independent of their spin multiplicity $(2S+1)$. A state $^(2S+1)\{\lambda\}$ will have

$$E_\lambda = |\lambda|\Delta E \tag{30}$$

For our four-fermion example the groundstate involves the 15 states arising from ${}^1\{2\} +^3 \{1^2\}$. The 6 states coming from ${}^1\{2\}$ can be viewed as arising from putting 2 fermions in the $1s$ orbitals and 2 in $1p$ orbitals to give rise to the two spectroscopic terms ${}^1SD$ while the 9 states coming from ${}^3\{1^2\}$ can likewise be viewed as arising from putting 2 fermions in the $1s$ orbitals and 2 in $1p$ orbitals but this time the orbital space is antisymmetric and the spin space symmetric and hence forming the spectroscopic term ^3P. NB. In atomic spectroscopy the one-electron states are given the traditional $n\ell$ labels viz.

$$1s, 2s, 3s, \ldots, 2p, 3p, 4p, \ldots, 3d, 4d, 5d, \ldots, 4f, 5f, 6f, \ldots \tag{31}$$

whereas in nuclear shell theory, where the isotropic three-dimensional harmonic oscillator is a useful starting point, the convention is to label the one-nucleon orbits with

$$1s, 2s, 3s, \ldots, 1p, 2p, 3p, \ldots, 1d, 2d, 3d, \ldots, 1f, 2f, 3f, \ldots \tag{32}$$

Here we follow the latter convention.

The next level involves the terms

$$^1(2\{21\} + \{3\}) +^3 (\{1^3\} + 2\{21\} + \{3\}) +^5 \{1^3\} \tag{31}$$

and hence a total degeneracy of 112. These are just the spectroscopic terms arising from the configurations $(1s)^2(1p)(2s),\ (1s)^2(1p)(1d),\ (1s)(1p)^3$.

In the above we have indicated how to count terms etc without calling upon the properties of the non-compact groups. That topic is largely covered in

K Grudziński and B G Wybourne, *Symplectic models of $n-$particle systems*, Rep. Math. Phys. **38**, 251-266 (1996).

Symmetric Functions and the Symmetric Group 12
B. G. Wybourne

12.1 Introduction

In this lecture I want to continue to discuss $N-$noninteracting fermions or bosons in a isotropic d-dimensional harmonic oscillator leading up to partition functions for such systems. This is of course just the beginning as one must eventually put in realistic interactions taking the non-interacting case as a basis. We want to pay particular attention to the enumeration of the *complete* set of basis states. This can be done in a number of ways each related to the other by some unitary transformation. One can start by considering the non-compact metapletic group $Mp(2Nd)$ and then working down through various compact and non-compact subgroups as in

1. K Grudziński and B G Wybourne, *Symplectic models of $n-$particle systems*, Rep. Math. Phys. **38**, 251-66.

However, in this lecture I shall try to keep the approach relatively simple, starting with a single fermion or boson in a isotropic d-dimensional harmonic oscillator to establish a basis and then discuss the case of various approaches to the problem of N identical noninteracting bosons or fermions. We will assume that the spin of the single particle is s_b (or s_f) for the boson (or the fermion). I shall assume familiarity with earlier lectures in this series.

12.2 Single-particle states for a isotropic d-dimensional harmonic oscillator

We introduce three schemes for describing a single particle (fermion or boson) in a isotropic d-dimensional harmonic oscillator.

1. A non-compact scheme

The infinite set of spatial states span the basic infinite dimensional unitary harmonic series irreducible representation $\tilde{\Delta}$ and classify the states under the scheme

$$SU(2) \times (Mp(2d) \supset Sp(2d, \Re) \supset U(d) \supset O(d) \supset \ldots U(1)) \tag{1}$$

Note that we have a direct product with $SU(2)$ being the group describing the spin part of our wavefunction and the $Mp(2d)$ group and its subgroups the spatial part. Recalling that under $Mp(2d) \supset Sp(2d, \Re)$

$$\tilde{\Delta} \to \Delta_+ + \Delta_- \tag{2}$$

while under $Sp(2d, \Re) \supset U(d)$

$$\Delta_+ \to M_+ \tag{3a}$$

$$\Delta_- \to M_- \tag{3b}$$

with

$$M_+ = \sum_{m=0}^{\infty} \{2m\} \tag{4a}$$

$$M_- = \sum_{m=0}^{\infty} \{2m+1\} \tag{4b}$$

$$M = M_+ + M_- = \sum_{m=0}^{\infty} \{m\} \tag{4c}$$

We also recall that under $U(d) \supset O(d)$ we have the general result

$$\{\lambda\} \to [\lambda/D] \tag{5}$$

where D is the infinite $S-$function series

$$D = \sum_{\delta}^{\infty} \{\delta\} \tag{6}$$

Where the summation is over all partitions (δ) whose parts are all *even*.

For details see

2. K Grudziński and B G Wybourne, Plethysm for the noncompact group Sp(2n,R) and new S-function identities J Phys A:Math.Gen.**29**, 6631-41 (1996).
3. R C King and B G Wybourne, *Holomorphic discrete series* ... J Phys A:Math.Gen.**18**, 3113-39 (1985).
4. R C King and B G Wybourne, *Products and symmetrized powers of irreducible representations of* $Sp(2n,\Re)$ *and their associates* J Phys A:Math.Gen.**31**, 6669-89 (1998).
5. R C King and B G Wybourne, *Analogies between finite-dimensional irreps of* $SO(2n)$ *and infinite-dimensional irreps of* $Sp(2n,\Re)$, J. Math. Phys.**41**, 5002-19 (2000).
6. R C King and B G Wybourne, *Analogies between finite-dimensional irreps of* $SO(2n)$ *and infinite-dimensional irreps of* $Sp(2n,\Re)$*, Part II:Plethysms,* J. Math. Phys. **41**, 5656-90 (2000).

■ 2. The $SU(2)\times U(d)$ scheme

In this scheme the spin s belongs to the group $SU(2)$ and spans the $SU(2)$ irreducible representation $\{2s\}$ while the spatial parts span the infinite set of irreducible representations of $U(d)$ labelled by one-part partitions $\{m\}$ so we can symbolically designate the $SU(2)\times U(d)$ single particle states by

$$\{2s\}\times M=\sum_{m=0}^{\infty}\{2s\}\times\{m\} \tag{7}$$

the distinction between bosons and fermions being made at the $SU(2)$ level. The *even* parity states will be associated with the *even* values of m and the *odd* parity states with the *odd* values of m.

■ 3. The $U(1)\times U(d)$ scheme

In this scheme we work at the spin projection level where the different m_s states span one-dimensional irreducible representations of the Abelian group $U(1)$ which we will choose to label as $\{m_s\}$ and remember that for $U(1)$ the Kronecker products are such that

$$\{p\}\times\{q\}=\{p+q\} \tag{8a}$$

while for symmetrized powers (or plethysms)

$$\{p\}\otimes\{\lambda\}=\begin{cases}0 & \text{if } \ell(\lambda)>1\\ p\times\lambda_1 & \text{if } \ell(\lambda)=1\end{cases} \tag{8b}$$

the complete set of single particle states will span the reducible representation

$$\sum_{m_s=-s}^{m_s=s}\{m_s\}\times M \tag{9}$$

■ 12.3 $N-$noninteracting particles in a isotropic d-dimensional harmonic oscillator

The distinction between bosons and fermions becomes crucial when we consider more than one particle. Throughout we shall assume that the N particles are indistinguishable. The basic ansatz is that for bosons the $N-$particle wavefunctions must be totally *symmetric* with respect to all permutations of the N particles while for fermions the $N-$particle wavefunctions must be totally *antisymmetric* with respect to all permutations of the N particles. In other words boson wavefunctions are permanental while those of fermions are determinantal. If our wavefunction is constructed as products of spin and spatial parts then the symmetrization of the spin and spatial parts need not themselves be symmetric (or antisymmetric) but their product must follow the correct statistics. Before continuing a brief diversion to recall some results involving plethysms, recalling parts of earlier lectures. Those unfamiliar with the properties of plethysms might consult some of the references listed in my publications, particularly publications 32,35,39,45,83,88,154 and references contained therein.

■ Plethysm for direct products of groups

In many applications we are involved with the direct product of two groups (more than two poses no new difficulties) say, $\mathcal{G}\times\mathcal{G}'$ with irreducible representations $A_{\mathcal{G}}\times B_{\mathcal{G}'}$ and we need to determine the

$\mathcal{G}\times\mathcal{G}'$ content of $N-$fold product of an irreducible representation say $(A\times B)^{\times N}$ (henceforth we drop the subscripts) and extract the part of the product symmetrized according to the permutational symmetry $\{\lambda\}$. In terms of plethysm we have

$$(A\times B)\otimes\{\lambda\}=\sum_{\rho}(A\otimes\{\rho\cdot\lambda\})\times(B\otimes\{\rho\}) \tag{10}$$

where $\{\rho\cdot\lambda\}$ signifies a $S-$function inner product which is null unless the partitions (ρ) and (λ) are of the same weight, i.e. $|\rho|=|\lambda|$. Two special cases are of interest

$$\{\rho\cdot\{\lambda\}=\begin{cases}\{\rho\} & \text{if } \{\lambda\}=\{N\} \text{ and } |\rho|=|\lambda|\\ \{\rho'\} & \text{if } \{\lambda\}=\{1^N\} \text{ and } |\rho|=|\lambda|\end{cases} \tag{11}$$

where the partition (ρ') is conjugate to (ρ).

By way of example we have

$$\begin{aligned}(A\times B)\otimes\{4\}=&(A\otimes\{4\})\times(B\otimes\{4\})+(A\otimes\{31\})\times(B\otimes\{31\})+(A\otimes\{2^2\})\times(B\otimes\{2^2\})\\&+(A\otimes\{21^2\})\times(B\otimes\{21^2\})+(A\otimes\{1^4\})\times(B\otimes\{1^4\})\end{aligned} \tag{12a}$$

$$\begin{aligned}(A\times B)\otimes\{1^4\}=&(A\otimes\{4\})\times(B\otimes\{1^4\})+(A\otimes\{31\})\times(B\otimes\{21^2\})+(A\otimes\{2^2\})\times(B\otimes\{2^2\})\\&+(A\otimes\{21^2\})\times(B\otimes\{31\})+(A\otimes\{1^4\})\times(B\otimes\{4\})\end{aligned} \tag{12b}$$

In many cases of interest only some of the terms in the right-hand-side of (12) will be non-null. This is particularly the case when one of the groups is of low rank, e.g, $SU(2)$ or $U(1)$. To be specific, let us henceforth consider bosons of spin $s_b=1$ and fermions of spin $s_f=\frac{1}{2}$. In this case the boson spin spans the $\{2\}$ irreducible representation of $SU(2)$ while the fermion spin spans the $\{1\}$ irreducible representation of $SU(2)$. There is no difficulty in going to higher spin states.

■ **The $SU(2)\times Mp(2d)$ scheme**

In this scheme the single particle spans the representation $\{2s\}\times\tilde{\Delta}$ of $SU(2)\times Mp(2d)$ and for $N-$noninteracting particles we have

$$(\{2\}\times\tilde{\Delta})\otimes\{N\}=\sum_{\rho\vdash N}(\{2\}\otimes\{\rho\})\times(\tilde{\Delta}\otimes\{\rho\})\quad\text{for bosons} \tag{13a}$$

$$(\{1\}\times\tilde{\Delta})\otimes\{1^N\}=\sum_{\rho\vdash N}(\{1\}\otimes\{\rho'\})\times(\tilde{\Delta}\otimes\{\rho\})\quad\text{for fermions} \tag{13b}$$

Let us consider the evaluation of the $SU(2)$ plethysms, first for fermions and then for bosons.

We have noted earlier that for fermions of spin $s_f=\frac{1}{2}$ that $\{1\}\otimes\{\rho\}=\{\rho\}$ and that the partition (ρ) can involve at most two parts and in $SU(2)$ we have the irreducible representation equivalence

$$\{\rho_1,\rho_2\}\equiv\{\rho_1-\rho_2\} \tag{14}$$

leading to

$$(\{1\}\times\tilde{\Delta})\otimes\{1^N\}=\sum_{S=S_{min}}^{\frac{N}{2}}{}^{2S+1}(\tilde{\Delta}\otimes\{2^{\frac{N}{2}-S}1^{2S}\}) \tag{15}$$

$$S_{min}=\begin{cases}\frac{1}{2} & \text{if } N \text{ is odd}\\ 0 & \text{if } N \text{ is even}\end{cases} \tag{16}$$

Thus for $N=4$ fermions we have

$$(\{1\}\times\tilde{\Delta})\otimes\{1^4\}={}^5(\tilde{\Delta}\otimes\{1^4\})+{}^3(\tilde{\Delta}\otimes\{21^2\})+{}^1(\tilde{\Delta}\otimes\{2^2\}) \tag{17}$$

Recalling the isomorphisms between $SO(3)$ and its covering group $SU(2)$ we have under $SU(2)\sim SO(3)$ $\{2\}\sim[1]$ leading to

$$\{2\}\otimes\{\rho\}\sim[\rho/D] \tag{18}$$

The right-hand-side of (18) gives the spins for each partition (ρ) appearing in (13a). Furthermore, (ρ) can involve at most three non-zero parts and those involving three non-zero parts are equivalent to a partition with two or less parts via

$$\{\rho_1, \rho_2, \rho_3\} \equiv \{\rho_1 - \rho_3, \rho_2 - \rho_3\} \tag{19}$$

NB If $[\rho/D]$ leads to partitions involving more than one non-zero part then the $SO(3)$ modification rules need to be applied. Assuming (19) has been applied leaving a $SO(3)$ non-standard irreducible representation $[a, b]$ then

$$[a, b] \equiv \begin{cases} 0 & \text{if } b \geq 2 \\ \\ [a] & \text{if } b = 1 \end{cases} \tag{20}$$

with the above in mind we can use (13a) to give for four spin 1 bosons

$$(\{2\} \times \tilde{\Delta}) \otimes \{4\} =^{(9+5+1)} (\tilde{\Delta} \otimes \{4\}) +^{(7+5+3)} (\tilde{\Delta} \otimes \{31\}) +^{3} (\tilde{\Delta} \otimes \{21^2\}) +^{(5+1)} (\tilde{\Delta} \otimes \{2^2\}) \tag{21}$$

Where again the multiplicities $(2S + 1)$ are given as left superscripts. To complete the examples of this scheme one should evaluate the various plethysms for the relevant metapletic group and then branch through the various subgroups. We shall not do that at this time.

- **The $SU(2) \times U(d)$ scheme**

In this scheme one starts with (7) and evaluates the relevant plethysms as in the previous scheme. For the spin part there are no changes. The $U(d)$ irreducible representations are combined as the single infinite dimensional reducible representation M. Thus for N spin $\frac{1}{2}$ fermions we have from noting (15)

$$(\{1\} \times M) \otimes \{1^N\} = \sum_{S=S_{min}}^{\frac{N}{2}} {}^{2S+1}(M \otimes \{2^{\frac{N}{2}-S}1^{2S}\}) \tag{22}$$

and for four fermions

$$(\{1\} \times M) \otimes \{1^4\} =^5 (M \otimes \{1^4\}) +^3 (M \otimes \{21^2\}) +^1 (M \otimes \{2^2\}) \tag{23}$$

as indeed found and expanded in the previous lecture. For N bosons of spin 1 the result comes from (21) by simply replacing $\tilde{\Delta}$ by M throughout.

- **The $U(1) \times U(d)$ scheme**

In this scheme we treat spin at the level of its projection m_s. Clearly in each scheme there must be a complete accounting of all the quantum states and respecting symmetrization. In the case of fermions of spin $\frac{1}{2}$ we have for N particles

$$\big((\{\tfrac{1}{2}\} \times M) + (\{-\tfrac{1}{2}\} \times M)\big) \otimes \{1^N\} = \sum_{x=0}^{N} \big((\{\tfrac{1}{2}\} \times M) \otimes \{1^{N-x}\}\big) \times \big((\{-\tfrac{1}{2}\} \times M) \otimes \{1^x\}\big) \tag{24}$$

Noting (8a) and (8b), we can rewrite (24) as

$$\big((\{\tfrac{1}{2}\} \times M) + (\{-\tfrac{1}{2}\} \times M)\big) \otimes \{1^N\} = \sum_{x=0}^{N} \big(\{\tfrac{N-x}{2}\} \times (M \otimes \{1^{N-x}\})\big) \times \big(\{-\tfrac{x}{2}\} \times (M \otimes \{1^x\})\big) \tag{25}$$

Notice that (25) involves the product of two terms, the first term, $\big(\{\frac{N-x}{2}\} \times (M \otimes \{1^{N-x}\})\big)$, involves states with spin projection $M_S = \frac{N-x}{2}$ (spin-up) which are antisymmetric in their spatial part while the second term, $\big(\{-\frac{x}{2}\} \times (M \otimes \{1^x\})\big)$,involves states with spin projection $M_S = -\frac{x}{2}$ (spin-down) which are again antisymmetric in their spatial part. Equation (25) involves Kronecker products in $U(1)$ and in $U(d)$ and (25) may be rearranged as

$$\big((\{\tfrac{1}{2}\} \times M) + (\{-\tfrac{1}{2}\} \times M)\big) \otimes \{1^N\} = \sum_{x=0}^{N} \big(\{\tfrac{N-x}{2}\} \times \{-\tfrac{x}{2}\}\big) \times \big((M \otimes \{1^{N-x}\}) \times (M \otimes \{1^x\})\big) \tag{26}$$

The first Kronecker product can be evaluated using (8a) to give

$$\big(\{\tfrac{N-x}{2}\} \times \{-\tfrac{x}{2}\}\big) = \{\tfrac{N}{2} - x\} \tag{27}$$

and the second using the plethysm property

$$(A \otimes \{\lambda\}) \times (A \otimes \{\mu\}) = A \otimes (\{\lambda\} \times \{\mu\}) \tag{28}$$

leading to

$$\big((M \otimes \{1^{N-x}\}) \times (M \otimes \{1^x\})\big) = M \otimes (\{1^{N-x}\} \cdot \{1^x\}) \tag{29}$$

with the $\cdot$ implying ordinary $S-$function multiplication. Combining (27) and (29) in (26) finally gives

$$\big((\{\tfrac{1}{2}\} \times M) + (\{-\tfrac{1}{2}\} \times M)\big) \otimes \{1^N\} = \sum_{x=0}^{N} \{\tfrac{N}{2} - x\} \times \big(M \otimes (\{1^{N-x}\} \cdot \{1^x\})\big) \tag{30}$$

For four fermions of spin $\frac{1}{2}$ we obtain

$$\begin{aligned}&\big((\{\tfrac{1}{2}\} \times M) + (\{-\tfrac{1}{2}\} \times M)\big) \otimes \{1^4\}\\ &= \{2\} \times (M \otimes (\{1^4\} \cdot \{0\})) + \{1\} \times (M \otimes (\{1^3\} \cdot \{1\})) + \{0\} \times (M \otimes (\{1^2\} \cdot \{1^2\}))\\ &+ \{-1\} \times (M \otimes (\{1\} \cdot \{1^3\})) + \{-2\} \times (M \otimes (\{0\} \cdot \{1^4\}))\end{aligned} \tag{31a}$$

$$\begin{aligned}&= (\{2\} + \{-2\}) \times (M \otimes \{1^4\}) + (\{1\} + \{-1\}) \times (M \otimes (\{1^3\} \cdot \{1\}))\\ &+ \{0\} \times (M \otimes (\{1^2\} \cdot \{1^2\}))\end{aligned} \tag{31b}$$

$$\begin{aligned}&= (\{2\} + \{-2\}) \times (M \otimes \{1^4\}) + (\{1\} + \{-1\}) \times (M \otimes (\{1^4\} + \{21^2\}))\\ &+ \{0\} \times (M \otimes (\{1^4\} + \{21^2\} + \{2^2\}))\end{aligned} \tag{31c}$$

Comparison with (17) and (23) shows, as should be, that the same number of quantum states are obtained in each scheme. We note that the above scheme was first used by Shudeman[6] to determine the states arising from configurations of equivalent electrons ℓ^N though without using group theory. It was then used by Judd[7] to recast atomic shell theory, Judd giving a group formulation to the scheme and naming it $LL-$coupling. I have given further details[9].

Let us return to the spin 1 bosons. Each boson has three spin states ($M_S = 0, \pm 1$) that can be described by the $U(1)$ irreducible representations $\{1\}, \{0\}, \{-1\}$. For $N-$noninteracting bosons we have from plethysm

$$\begin{aligned}&(\{1\} \times M + \{0\} \times M + \{-1\} \times M) \otimes \{N\}\\ &= \sum_{x=0}^{N}\sum_{y=0}^{x} [((\{1\} \times M) \otimes \{N-x\}) \times ((\{0\} \times M) \otimes \{x-y\}) \times ((\{-1\} \times M) \otimes \{y\})]\end{aligned} \tag{32a}$$

$$= \sum_{x=0}^{N}\sum_{y=0}^{x} [(\{N-x\} \times (M \otimes \{N-x\})) \times (\{0\} \times (M \otimes \{x-y\})) \times (\{-y\} \times (M \otimes \{y\}))] \tag{32b}$$

$$= \sum_{x=0}^{N}\sum_{y=0}^{x} [\{N-x-y\} \times (M \otimes (\{N-x\} \cdot \{x-y\} \cdot \{y\})] \tag{32c}$$

where in (32c) the spin projection quantum number, M_S is

$$M_S = N - x - y \tag{33}$$

For brevity, let us define

$$M_S^{\uparrow\downarrow}(S) = \begin{cases} \{S\} + \{-S\} & \text{if } S > 0 \\ \{0\} & \text{if } S = 0 \end{cases} \tag{34}$$

For four spin 1 bosons we have from (32c)

$$
\begin{aligned}
&(\{1\}\times M+\{0\}\times M+\{-1\}\times M)\otimes\{4\}\\
&=M_S^{\uparrow\downarrow}(4)(M\otimes\{4\})+M_S^{\uparrow\downarrow}(3)(M\otimes\{3\}\cdot\{1\})+M_S^{\uparrow\downarrow}(2)(M\otimes(\{3\}\cdot\{1\}+\{2\}\cdot\{2\})\\
&+M_S^{\uparrow\downarrow}(1)(M\otimes(\{2\}\cdot\{1\}\cdot\{1\}+\{3\}\cdot\{1\})+M_S^{\uparrow\downarrow}(0)(M\otimes(\{4\}+\{2\}\cdot\{2\}+\{2\}\cdot\{1\}\cdot\{1\})
\end{aligned}
\tag{35a}
$$

$$
\begin{aligned}
&=M_S^{\uparrow\downarrow}(4)(M\otimes\{4\})+M_S^{\uparrow\downarrow}(3)(M\otimes(\{4\}+\{31\}))+M_S^{\uparrow\downarrow}(2)(M\otimes(2\{4\}+2\{31\}+\{2^2\}))\\
&+M_S^{\uparrow\downarrow}(1)(M\otimes(2\{4\}+3\{31\}+\{2^2\}+\{21^2\}))+M_S^{\uparrow\downarrow}(0)(M\otimes(3\{4\}+3\{31\}+2\{2^2\}+\{21^2\}))
\end{aligned}
\tag{35b}
$$

which is consistent with the M_S projection of the spins found in (21).

I am indebted to Jürgen Schnack for pointing out to me the relevance of the scheme for computing partition functions.

7. C L B Shudeman, J. Franklin Inst. **224**, 501 (1937).

8. B R Judd, *Atomic Shell Theory Recast*, Phys. Rev. **162**, 28-37 (1967).

9. B G Wybourne, *Coefficients of fractional parentage and LL-Coupling*, J. de Phys. **30**, 35-8 (1969).

Additional information on boson-fermion relationships, not covered in these lectures, may be found in

10. B G Wybourne, *Hermite's Reciprocity Law and the Angular Momentum States of Equivalent Particle Configurations*, J. Math. Phys. **10**, 467-71 (1969).

11. B G Wybourne, *Statistical and Group Properties of the Fractional Quantum Hall Effect* (SSPCM'2000, 31 August - 6 September 2000, Myczkowce, Poland) Singapore: World Scientific (In Press).