stattrek.com/statistics/dictionary.aspx?definition=Central limit theorem

Statistics and Probability Dictionary

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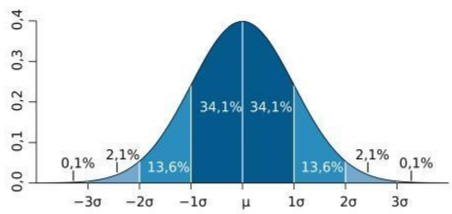
Select term: Central Limit Theorem

Central Limit Theorem

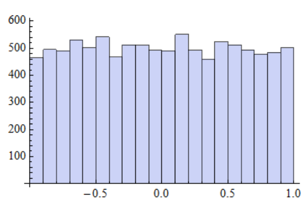
The **central limit theorem** states that the sampling distribution of the mean of any independent, random variable will be normal or nearly normal, if the sample size is large enough.



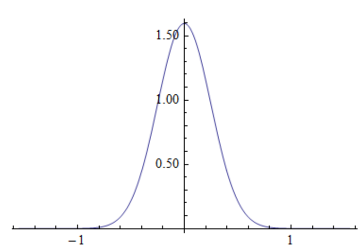
$$f(x;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

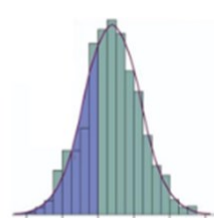
Distribució Gaussiana

```
Histogram[RandomReal[{-1, 1}, 10 000]]
```

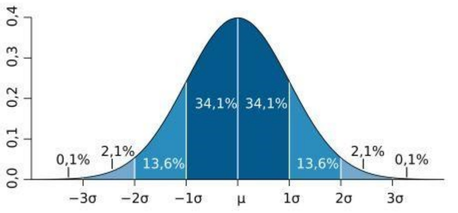



```
n = 1000;
data = Table[0, {i, 1, n}];
For[i = 1, i ≤ n, i++,
  data[[i]] = Mean[RandomReal[{-1, 1}, 10 000]]];
SmoothHistogram[data, 0.25]
```





Distribució Gaussiana



$\mu = \overline{x} \pm 2\,\sigma$

Mesures indirectes $y = f(x)$

$$\bar{y} = \frac{1}{N}\sum_i f(x_i) = \frac{1}{N}\sum_i f(\bar{x} + \delta_i) \cong \frac{1}{N}\sum_i \left[f(\bar{x}) + \delta_i \left(\frac{df}{dx}\right)\right] = f(\bar{x}) + \frac{1}{N}\sum_i f(\delta_i) = f(\bar{x})$$

Propagació de la imprecisió

$$y = f(x) \quad \rightarrow \quad dy = \left(\frac{df}{dx}\right) dx \quad \rightarrow \quad (y - y_i)^2 = \left(\frac{df}{dx}\right)^2 (x - x_i)^2$$

$$\sum_i (y - y_i)^2 = \left(\frac{df}{dx}\right)^2 \sum_i (x - x_i)^2 \quad \rightarrow \quad \boxed{S_y^2 = \left(\frac{df}{dx}\right)^2 S_x^2}$$

Propagació de la imprecisió: dos variables

$$z = f(x, y) \quad \rightarrow \quad dz = \left(\frac{\partial f}{dx}\right) dx + \left(\frac{\partial f}{dy}\right) dy$$

$$\rightarrow \quad (z - z_{ij}) = \left(\frac{\partial f}{dx}\right)(x - x_i) + \left(\frac{\partial f}{dy}\right)(y - y_j)$$

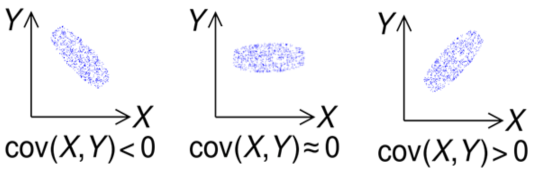
$$\rightarrow \quad \frac{1}{N^2}\sum_{i,j}(z - z_{ij})^2 = \frac{1}{N^2}\left(\frac{\partial f}{dx}\right)^2 N \sum_i (x - x_i)^2 + \frac{1}{N^2}\left(\frac{\partial f}{dy}\right)^2 N \sum_j (y - y_j)^2 + \frac{2}{N^2}\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial f}{\partial y}\right)\sum_i (x - x_i)\sum_j (y - y_j)$$

$$\boxed{S_z^2 = \left(\frac{df}{dx}\right)^2 S_x^2 + \left(\frac{df}{dy}\right)^2 S_y^2 + 2\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial f}{\partial y}\right) S_{xy}^2}$$

x,y indep. $\rightarrow S_{xy}^2 = 0$

$$\boxed{S_f^2 = \sum_i \left(\frac{\partial f}{\partial x_i}\right)_{x_j}^2 S_{x_i}^2}$$

$$\frac{1}{N^2}\sum_{ij}(\bar{x}-x_i)(\bar{y}-y_j) = \frac{1}{N^2}\sum_{ij}x_i y_j - \frac{1}{N}\,\bar{y}\sum_i x_i - \frac{1}{N}\,\bar{x}\sum_j y_i + \bar{x}\,\bar{y} = (\overline{xy} - \bar{x}\,\bar{y})$$



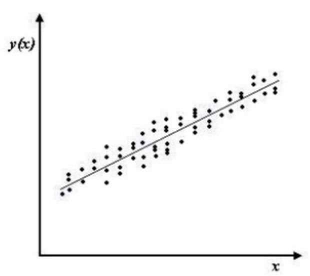
Imprecisió de la mitjana

$$\bar{x} = \frac{1}{N}\sum_i x_i \quad \rightarrow \quad \bar{x} = f(x_1, x_2, \ldots, x_N)$$

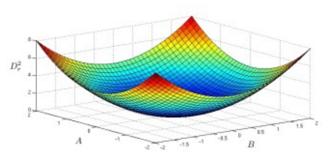
$$\frac{\partial \bar{x}}{\partial x_i} = \frac{1}{N} \quad \rightarrow \quad S_{\bar{x}}^2 = \frac{1}{N^2}\sum_i S_x^2 = \frac{1}{N^2} N S_x^2 \quad \rightarrow \quad \boxed{S_{\bar{x}}^2 = \frac{S_x^2}{N}}$$

Ajust Lineal $\quad y = A + Bx$

x	y
x_1	y_{11}
x_1	y_{12}
x_2	y_{21}
x_3	y_{31}
x_3	y_{32}
x_3	y_{33}
$\vdots$	$\vdots$



$$D_r^2 = \sum_{ij}(y_{ij} - y_i^c)^2 = \sum_{ij}(y_{ij} - A - Bx_i)^2$$



$$\left.\begin{aligned} \frac{\partial D_r^2}{\partial A} &= 0 = -2\sum_{ij}(y_{ij} - A - Bx_i) \rightarrow \bar{y} = A + B\bar{x} \\ \frac{\partial D_r^2}{\partial B} &= 0 = -2\sum_{ij}(y_{ij} - A - Bx_i)x_i \rightarrow \overline{xy} = A\bar{x} + B\overline{x^2} \end{aligned}\right\} \quad \overline{xy} = (\bar{y} - B\bar{x})\bar{x} + B\overline{x^2}$$

$$\boxed{B = \frac{\overline{xy} - \bar{x}\bar{y}}{\overline{x^2} - \bar{x}^2}} \qquad \boxed{A = \bar{y} - B\bar{x}}$$

$$\boxed{S_y^2 = \sum_i \left(\frac{df}{dx_i}\right)^2 S_{x_i}^2}$$

Càlcul S^2_B

$$B = \frac{\overline{xy} - \bar{x}\bar{y}}{\overline{x^2} - \bar{x}^2} = \frac{1}{N}\frac{\sum_{ij} y_{ij}x_i - \bar{x}y_{ij}}{\overline{x^2} - \bar{x}^2} \longrightarrow \frac{\partial B}{\partial y_{ij}} = \frac{1}{N}\frac{x_i - \bar{x}}{\overline{x^2} - \bar{x}^2}$$

$$S_B^2 = \frac{1}{N}\frac{1}{(\overline{x^2} - \bar{x}^2)^{\cancel{2}}}\cancel{\frac{1}{N}\sum_{ij}(x_i - \bar{x})^2}\, S_r^2 \longrightarrow \boxed{S_B^2 = \frac{1}{N}\frac{1}{\overline{x^2} - \bar{x}^2}\, S_r^2}$$

Càlcul S^2_A $\quad A = \bar{y} - B\bar{x} = \sum_{ij}\left(\frac{1}{N} - \frac{\bar{x}\,(x_i - \bar{x})}{D_x^2}\right)y_{ij} \longrightarrow \boxed{S_A^2 = \left(\frac{1}{N} + \frac{\bar{x}^2}{D_x^2}\right)S_{reg}^2 = \frac{\overline{x^2}}{D_x^2}\; S_{reg}^2}$

Càlcul S^2_{yc} $\quad y_i^c = \bar{y} - B(x_i - \bar{x}) = \frac{1}{N}\sum_{ij}\left(1 - \frac{1}{\overline{x^2} - \bar{x}^2}(x_i - \bar{x})\right)y_{ij}$

$$\longrightarrow \boxed{S_{y_i^c}^2 = \frac{S_{reg}^2}{N}\left[1 + \frac{N(x_i - \bar{x})^2}{D_x^2}\right]}$$

Altres ajustos lineals $\quad y = A_0 + Bx$

$$D_r^2 = \sum_{ij}(y_{ij} - y_i^c)^2 = \sum_{ij}(y_{ij} - A_0 - Bx_i)^2$$

$$\frac{dD_r^2}{dB} = 0 = -2\sum_{ij}(y_{ij} - A_0 - Bx_i)x_i \quad \rightarrow \overline{xy} - A_0\bar{x} - B\overline{x^2} = 0$$

$$\boxed{B = \frac{\overline{xy} - A_0\bar{x}}{\overline{x^2}}}$$

Càlcul S^2_B $\qquad$ recordem que: $\boxed{S_y^2 = \sum_i \left(\frac{df}{dx_i}\right)^2 S_{x_i}^2}$

$$\frac{\partial B}{\partial y_{ij}} = \frac{1}{N}\frac{x_i}{\overline{x^2}} \longrightarrow \boxed{S_B^2 = \frac{1}{N}\frac{1}{\overline{x^2}}\, S_r^2}$$

etc.

En forma matricial

$$y_i^c = a + bx_i \qquad [y_i^c] = [1 \;\; x_i] \cdot \begin{bmatrix} a \\ b \end{bmatrix} \qquad \begin{bmatrix} y_1^c \\ \dots \\ y_n^c \end{bmatrix} = \begin{bmatrix} 1 & x_1 \\ \dots & \dots \\ 1 & x_n \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\boxed{\underset{n\times 1}{Y} = \underset{n\times 2}{X} \cdot \underset{2\times 1}{A}}$$

$$Y = X\,A \longrightarrow X^T Y = X^T X\, A \longrightarrow (X^T X)^{-1} X^T Y = (X^T X)^{-1}\,(X^T X)\,A = A$$

$$\boxed{A = (X^T X)^{-1} X^T Y}$$

En forma matricial: imprecisions

$$S_z^2 = \left(\frac{\partial f}{\partial x}\right)_y^2 S_x^2 + \left(\frac{\partial f}{\partial y}\right)_x^2 S_y^2 + 2\left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial f}{\partial y}\right)_x S_{xy}^2 = \left[\left(\frac{\partial f}{\partial x}\right)_y \;\; \left(\frac{\partial f}{\partial y}\right)_x\right] \begin{pmatrix} S_x^2 & S_{xy}^2 \\ S_{yx}^2 & S_y^2 \end{pmatrix} \begin{bmatrix} \left(\frac{\partial f}{\partial x}\right)_y \\ \left(\frac{\partial f}{\partial y}\right)_x \end{bmatrix} = Z^T \cdot cov \cdot Z$$

$$y_i^c = a + bx_i \longrightarrow S_{y_i^c}^2 = Z_i^T \cdot cov \cdot Z_i \begin{cases} Z_i^T = \left[\left(\dfrac{\partial y_i^c}{\partial a}\right)_b \;\; \left(\dfrac{\partial y_i^c}{\partial b}\right)_a\right] = [1 \quad x_i] \\[2ex] cov = \begin{pmatrix} S_a^2 & S_{ab}^2 \\ S_{ba}^2 & S_b^2 \end{pmatrix} \end{cases}$$

$$\boxed{S_{y_i^c}^2 = Z_i^T \cdot cov \cdot Z_i}$$

$$\boxed{S^2_{y_i^c} = Z_i^T \cdot cov \cdot Z_i}$$

$$S^2_{y_i^c}\, Z_i \cdot Z_i^T = Z_i \cdot Z_i^T \cdot cov \cdot Z_i \cdot Z_i^T$$

$$S^2_{y_i^c}\left(Z_i \cdot Z_i^T\right)^{-1}\left(Z_i \cdot Z_i^T\right) = \left(Z_i \cdot Z_i^T\right)\left(Z_i \cdot Z_i^T\right)^{-1} \cdot cov \cdot Z_i \cdot Z_i^T$$

$$\boxed{S^2_{y_i^c} = cov \cdot Z_i \cdot Z_i^T}$$

$$\sum_i S^2_{y_i^c} = cov \cdot \sum_i \begin{bmatrix} 1 \\ x_i \end{bmatrix} \cdot \begin{bmatrix} 1 & x_i \end{bmatrix} = cov \cdot \sum_i \begin{pmatrix} 1 & x_i \\ x_i & x_i^2 \end{pmatrix} = cov \cdot \begin{pmatrix} n & \Sigma x_i \\ \Sigma x_i & \Sigma x_i^2 \end{pmatrix}$$

$$= cov \cdot \begin{bmatrix} 1 & \dots & 1 \\ x_1 & \dots & x_n \end{bmatrix} \cdot \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} = cov \cdot Z^T \cdot Z$$

$$\boxed{\sum_i S^2_{y_i^c} = cov \cdot Z^T \cdot Z}$$

$$\boxed{\sum_i S^2_{y_i^c} = cov \cdot Z^T \cdot Z}$$

$$\Sigma_i S^2_{y_i^c} =? \quad \left\{ \begin{array}{l} S^2_{\bar{x}} = \frac{1}{n} S^2_x \\ S^2_{y_i^c} = \frac{1}{n} S^2_{y_i} \\ \sum_i S^2_{y_i^c} = \frac{1}{n} \sum_i S^2_{y_i} = S^2_r \end{array} \right\} \quad \sum_i S^2_{y_i^c} = S^2_r$$

$$\longrightarrow \quad S^2_r = cov \cdot \left(Z^T \cdot Z\right) \rightarrow \quad \boxed{cov = \left(Z^T \cdot Z\right)^{-1} S^2_r}$$

$$\boxed{S^2_{y_i^c} = Z_i^T \cdot cov \cdot Z_i = Z_i^T \cdot \left(Z^T \cdot Z\right)^{-1} \cdot Z_i \; S^2_r}$$

EXCEL

Regressió Lineal

X

1	x	y
1	1	2,2
1	2	3,3
1	3	3,8
1	4	5,2
1	5	6,2
1	6	6,8
1	7	8,1
1	8	9,3
1	9	9,9
1	10	10,6

$$Y = X\,A \qquad X = \begin{bmatrix} 1 & x_1 \\ \dots & \dots \\ 1 & x_n \end{bmatrix} \quad A = \begin{bmatrix} a \\ b \end{bmatrix} \longrightarrow A = (X^T X)^{-1} X^T Y$$

$x^T x$	10	55	
	55	385	
MMULT(TRANSPOSA(B12:C21);B12:C21)			

$(x^T x)^{-1}$	0,466667	-0,06667
	-0,06667	0,012121
MINVERSA(C24:D25)		

$x^T y$	65,4		
	439		
MMULT(TRANSPOSA(B12:C21);D12:D21)			

$(x^T x)^{-1} x^T y=$	a	1,253333
	b	0,961212
MMULT(C28:D29;G24:G25)		

y_c	$(y-y_c)^2$
2,214545	0,00021157
3,175758	0,01543618
4,13697	0,113548577
5,098182	0,010366942
6,059394	0,019770064
7,020606	0,048667034
7,981818	0,013966942
8,94303	0,127427365
9,904242	1,79982E-05
10,86545	0,070466116
Sr^2	0,052484848

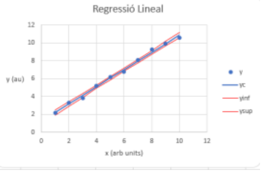
$$cov = (X^T X)^{-1} S_r^2$$

cov	0,024493	-0,0035
	-0,0035	0,000636

cov	=J22*sr_2	=K22*sr_2
	=J23*sr_2	=K23*sr_2

$$S^2_{y_i^c} = (1 \quad x_i)\; cov \begin{pmatrix} 1 \\ x_i \end{pmatrix}$$

Syc
0,1346519
0,1142002
0,0960448
0,0817306
0,0735359
0,0735359
0,0817306
0,0960448
0,1142002
0,1346519



Regressió no Lineal $y = f(\underline{x}, \underline{\theta})$, $Z_{iu}^0 = \left(\frac{\partial f(\underline{x_u}, \underline{\theta})}{\partial \theta_i}\right)$

$$\boxed{cov = (Z_0^T Z_0)^{-1} S_r^2}$$

$$\boxed{S_{y_i^c}^2 = Z_i^T cov\, Z_i} \quad Z_i = \begin{pmatrix} Z_{i1}^0 & \dots & Z_{ip}^0 \end{pmatrix}$$

EXCEL

Regressió no Lineal

X(mL/min)	Y(mm)
3,4	9,59
7,1	5,29
16,1	3,63
20	3,42
23,1	3,46
34,4	3,06
40	3,25
44,7	3,31
65,9	3,5
78,9	3,86
96,8	4,24

Eq. Teòrica	
Y=A X+ B/X+C	
A	1
B	1
C	1

Solver

A	0,024808
B	26,80485
C	1,548675

Y teorica	(Y-Yc)^2
9,52	0,01
5,50	0,04
3,61	0,00
3,39	0,00
3,28	0,03
3,18	0,01
3,21	0,00
3,26	0,00
3,59	0,01
3,85	0,00
4,23	0,00
SUMA	0,110191648
Sr2	0,013773956

$$cov = (Z_0^T Z_0)^{-1} S_r^2$$

$$\underbrace{\left(\frac{\partial f(\underline{x_u}, \underline{\theta})}{\partial \theta_i}\right)_{\underline{\theta_0}}}_{Z_{iu}^0}$$

∂Y/∂A = x	∂Y/∂B= 1/x	∂Y/∂C=1
3,4	0,294117647	1
7,1	0,14084507	1
16,1	0,062111801	1
20	0,05	1
23,1	0,043290043	1
34,4	0,029069767	1
40	0,025	1
44,7	0,022371365	1
65,9	0,015174507	1
78,9	0,012674271	1
96,8	0,010330579	1

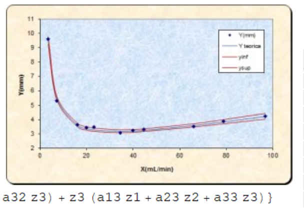
Z_i^T

Z^TZ	25974,5	11	430,4
	11	0,11704258	0,704985051
	430,4	0,70498505	11

$(Z^T Z)^{-1}$	0,00018844	0,04348822	-0,010160202
	0,04348822	23,9522	-3,236661149
	-0,0101602	-3,23666115	0,695886239

$\times S_r^2 \leftarrow cov$

parametre	estimació	S^2
A	0,02480846	0,00034171
B	26,8048521	0,36920885
C	1,54867494	0,02133138



Syc
0,11034715
0,05397389
0,05181047
0,05089314
0,04960845
0,04373915
0,04144147
0,04035411
0,04830205
0,06148399
0,08440633

$$S_{y_i^c}^2 = Z_i^T cov\, Z_i$$

$$(z1\ z2\ z3)\begin{pmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{pmatrix}\begin{pmatrix} z1 \\ z2 \\ z3 \end{pmatrix} = \{z1\,(a11\,z1 + a21\,z2 + a31\,z3) + z2\,(a12\,z1 + a22\,z2 + a32\,z3) + z3\,(a13\,z1 + a23\,z2 + a33\,z3)\}$$